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A CONTINUOUS DEFORMATION ALGORITHM ON THE PRODUCT SPACE OF UNIT SIMPLICES*†

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A continuous deformation algorithm is introduced on $S \times [1, \infty)$, where S denotes the product space of unit simplices, with arbitrary grid refinement between two subsequent levels. The set $S \times [1, \infty)$ is triangulated in such a way that for each m, $m = 1, 2, \ldots, S \times \{m\}$ is triangulated by the so-called V-triangulation. The algorithm starts by applying a variable dimension algorithm on S until an approximating simplex has been found on level 1. Then the algorithm follows a path of approximating simplices in $S \times [1, \infty)$, starting on level 1, until a certain level or a certain accuracy of a solution of the underlying problem has been reached. If the algorithm returns to level 1, then we again apply the variable dimension algorithm until a new approximating simplex is found on level 1, etc. We allow solutions to lie on the boundary of $S \times [1, \infty)$ in which case the algorithm, in general, will follow a path on the boundary of $S \times [1, \infty)$.

1. Introduction. To compute equilibria or fixed points on the unit simplex $S^n = \{x \in R^{n+1} \mid \sum_{i=1}^{n+1} x_i = 1,\ x_i \geqslant 0,\ i = 1, \ldots, n+1\}$ several simplicial algorithms have been developed. In a simplicial subdivision of S^n into n-dimensional simplices such an algorithm searches for an n-simplex which yields an approximate solution. If the approximation is not good enough the simplicial subdivision is refined in the hope that the approximate solution found for the new subdivision is better, etc. The so-called variable dimension restart simplicial algorithms can start anywhere and find for a given simplicial subdivision within a finite number of steps an approximate solution by generating a sequence of adjacent simplices of varying dimension of the simplicial subdivision. If necessary these algorithms can be restarted in or close to the last found approximation for a finer subdivision to find a better one. The several variable dimension restart simplicial algorithms developed thusfar differ from each other by the underlying triangulation or simplicial subdivision of S^n and the number of rays along which the arbitrarily chosen starting point can be left. Simplicial algorithms with $n + 1$ rays were introduced for the well known Q-triangulation of S^n in van der Laan and Talman [10], for the U-triangulation of the affine hull of S^n in van der Laan and Talman [11] and for the so-called V-triangulation of S^n in Doup and Talman [1]. Although the U-triangulation does not simplicially subdivide S^n itself this triangulation seems to be both in theory and in practice better than the Q-triangulation. The V-triangulation differs from both the U- and the Q-triangulations since it depends on the arbitrarily chosen starting point of the algorithm. In some way the V-triangulation is related to the K'-triangulation of R^n originally proposed in Todd [17]. An algorithm with $2^{n+1} - 2$ rays was recently proposed in Doup, van der Laan and Talman [2]. In

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this algorithm the V-triangulation underlies the algorithm. The other two triangulations do not seem to be appropriate for this algorithm with more than $n + 1$ rays.

In van der Laan and Talman [13] the $(n + 1)$-ray algorithm for both the Q- and U-triangulations has been generalized in order to compute equilibria or fixed points on the product space of several, say N, unit simplices S^{n_j}, $j = 1, \ldots, N$. This algorithm has $\Sigma_{j=1}^{N}(n_j + 1)$ rays to leave the arbitrarily chosen starting point in S, one ray to each facet of S. Recently, Doup and Talman [1] introduced such an algorithm on S for the V-triangulation, generalized for S, with $\Pi_{j=1}^{N}(n_j + 1)$ rays, one to each vertex of S. When applied for $N = 1$, $n_1 = n$, both algorithms simplify to the above-mentioned algorithms on S^n with $n + 1$ rays.

Instead of restarting a variable dimension simplicial algorithm on S^n, as soon as an approximating solution has been found, one can also continue the algorithm with the simplex yielding the approximating solution by embedding S^n into the set $S^n \times [1, \infty)$. This set is triangulated in such a way that for each m, $m = 1, 2, \ldots$, S^n is triangulated on level m with mesh tending to zero if m goes to infinity. In this way a path of adjacent $(n + 1)$-simplices of the triangulation of $S^n \times [1, \infty)$ is generated such that each generated simplex yields an approximate solution. Under some boundary condition, guaranteeing that the algorithm cannot terminate in the boundary of $S^n \times [1, \infty)$, such an algorithm will exceed each level m, $m = 1, 2, \ldots$, within a finite number of steps. As soon as some accuracy for the approximation is reached the algorithm can be stopped. Such algorithms are called homotopy or continuous deformation algorithms and were initiated in [4] for problems on S^n and in [5] on R^n. However, the triangulation used in the latter algorithm only allows for a grid refinement between two subsequent levels of at most two. Arbitrary grid refinement algorithms were developed in van der Laan and Talman [12] and Shamir [15] for the Q- and U-triangulations. Continuous deformation algorithms on the product space of more than one unit simplex are thus far unknown although both the Q- and the U-triangulations of S allow us to construct triangulations of $S \times [1, \infty)$. However the system of linear equations for the $\Sigma_{j=1}^{N}(n_j + 1)$-ray algorithm is not appropriate for continuation in $S \times [1, \infty)$ when an approximation has been found on level 1.

In this paper we will show how the recently developed variable dimension restart algorithm on S described in [1] can be adapted to a continuous deformation algorithm on $S \times [1, \infty)$ with arbitrary grid refinement between two subsequent levels. The triangulation of $S \times [1, \infty)$ which underlies the algorithm is based on the V-triangulation of S itself whereas the system of equations in which the l.p. pivot steps are made is similar to the system of equations for the restart algorithm. To start the algorithm the variable dimension restart algorithm of [1] is applied in order to find an approximating simplex in S on level 1. Then the algorithm generates a path of adjacent approximating simplices in $S \times [1, \infty)$ by alternating l.p. pivot steps and replacement steps in the triangulation. As soon as the algorithm returns to $S \times \{1\}$, the restart algorithm is again applied in order to find a new approximating simplex in S on level 1. Then the algorithm continues with this simplex in $S \times \{1\}$ generating again a path of adjacent approximating simplices in $S \times [1, \infty)$, etc. Since the number of simplices in $S \times [1, m]$ is finite for each m, $m = 1, 2, \ldots$, the algorithm must exceed each level m within a finite number of steps. The algorithm can be terminated when some accuracy is reached or a simplex on some specific level has been generated. Since we will not assume that the boundary condition holds for the underlying equilibrium or fixed point problem, we allow the algorithm to generate lower dimensional approximating simplices on the boundary of $S \times [1, \infty)$ so that the algorithm will generate in general a path of adjacent simplices of variable dimension. Restart algorithms on S which allow for these general type of problems were developed for the Q-triangulation in Freund [7] and in van der Laan, Talman and Van der Heyden [14], and for the V-triangulation in [1]. The first two algorithms cannot be adapted to a continuous deformation algorithm.

The advantage of a continuous deformation algorithm seems to be that as soon as an approximating simplex on say level m, $m \geqslant 1$, is found more information is used to find an approximating simplex on level $m+1$ when compared to a restart algorithm. More precisely, a restart algorithm only uses the information of the approximating solution whereas a continuation algorithm uses the information of the whole approximating simplex which includes the function values of the vertices of this simplex. Although this information might be of little value when the grid size of the triangulation is large, it could accelerate the algorithm considerably when the mesh becomes smaller, especially when the underlying problem is smooth so that a grid refinement factor of more than two can be taken between two subsequent levels.

The algorithm presented in this paper can be used to approximate Nash equilibria strategy vectors in an N-person noncooperative game. Then $S = \prod_{j=1}^{N} S^{n_j}$ is the strategy space of the game and S^{n_j}, $j = 1, \ldots, N$, is the strategy space of player j if $n_j + 1$ is the number of pure strategies of player j. Another application is the international trade model (see e.g. van der Laan [9]). Furthermore the homotopy parameter t, $t \geqslant 1$, in $S \times [1, \infty)$ can be considered as a time parameter. For example, the excess demand functions for the different goods in the international trade model might change continuously over time and we are interested in the path of equilibria considered as a function of time (see e.g. [6] and [8]). By triangulating $S \times [1, \infty)$ as described in this paper this path of solutions can be followed. For this application one in general does not need to refine the grid size on a new level. Although we have described the triangulation for a sequence of decreasing grid sizes on the subsequent levels, it will appear that the description of the triangulation is still valid if we take the same grid size on each level.

The paper is organized as follows. In §2 we describe the piecewise linear path of points in $S \times [1, \infty)$ followed by the new algorithm. §3 reviews the product-ray algorithm on S and describes therefore how the path is followed in $S \times \{1\}$. The triangulation of $S \times [1, \infty)$ and of its boundaries is treated in the §4 and 5 respectively. Finally, §6 gives the steps of the new algorithm in detail.

2. The path of points followed by the continuous deformation algorithm. Let S be the product space of N unit simplices, i.e. $S = \prod_{j=1}^{N} S^{n_j}$ where $S^{n_j} = \{ p_j \in R_+^{n_j+1} \mid \sum_{k=1}^{n_j+1} p_{j,k} = 1\}$. A vector p in S is denoted by $p = (p_1, \ldots, p_N)$ with $p_j = (p_{j,1}, \ldots, p_{j,n_j+1})^{\mathrm{T}}$ an element of S^{n_j}, $j \in I_N = \{1, \ldots, N\}$. Both the kth component of an element p_j in S^{n_j} and the $(\sum_{i=1}^{j-1}(n_i+1)+k)$th component of a vector p in S is therefore denoted by $p_{j,k}$, $k = 1, \ldots, n_j + 1$, $j \in I_N$. For $j = 1, \ldots, N$, the index set $\{(j,1), \ldots, (j, n_j+1)\}$ is denoted by $I(j)$, where (j,k) denotes the index $\sum_{i=1}^{j-1}(n_i + 1) + k$, $k = 1, \ldots, n_j + 1$. Moreover, the set I is the union of $I(j)$ over all $j \in I_N$. The (i,h)th unit vector in $\prod_{j=1}^{N} R^{n_j+1}$ is denoted by $e(i,h)$, $(i,h) \in I$, and we define $n = \sum_{j=1}^{N} n_j$.

Now let z be a continuous function from S to R^{N+n} satisfying for all p in S

$$p_j^{\mathrm{T}} z_j(p) = \sum_{k=1}^{n_j+1} p_{j,k} z_{j,k}(p) = 0 \quad \text{for all } j \in I_N. \tag{2.1}$$

The nonlinear complementarity problem on S with respect to z is to find a vector p^* in S such that $z(p^*) \leqq 0$. Observe that (2.1) implies that at such a solution, for all p, $p^{\mathrm{T}} z(p^*) \leqslant p^{*\mathrm{T}} z(p^*) = 0$ so that the nonlinear complementarity problem (NLCP) on S is to find an outward normal of z on S, see e.g. Eaves [3]. To solve the nonlinear complementarity problem on S we embed S in the set $S \times [1, \infty)$ and triangulate $S \times [1, \infty)$ in such a way that for an arbitrary sequence $k_1, k_2, \ldots$ of integers larger than one, the set $S \times \{m\}$ is triangulated according to the so-called V-triangulation of S with grid size $d_m^{-1} = k_{m-1}^{-1} d_{m-1}^{-1}$, $m = 2, 3, \ldots$, where d_1 is an arbitrary chosen positive integer, d_1^{-1} being the grid size of the V-triangulation of S on level 1. The V-triangulation of S is due to Doup and Talman [1] and is reviewed in §3. The V-triangulation of $S \times \{m\}$ with grid size d_m^{-1}, $m = 1, 2, \ldots$, induces a simplicial subdivision of $S \times [1, \infty)$ by triangulating for each m the cell $\tau^m \times [m, m+1]$ in an appropriate way for each n-dimensional simplex τ^m of the V-triangulation of S on level m. This triangulation of $S \times [1, \infty)$ is described in §4. Moreover, this triangulation induces a triangulation of the boundary faces $S(U) \times [1, \infty)$ where for $U \subset I$ the boundary face $S(U)$ of S is defined by

$$S(U) = \{x \in S | x_{j,k} = 0, (j, k) \in U\}.$$

The triangulation of $S(U) \times [1, \infty)$ is described in detail in §5. Let h be the continuous function from $S \times [1, \infty)$ to R^{N+n} defined by $h(p, t) = z(p)$ for all (p, t) in $S \times [1, \infty)$ and let g be the piecewise linear approximation of h with respect to the underlying triangulation of $S \times [1, \infty)$. More precisely, let $x = (p, t)$ be an arbitrary point in $S \times [1, \infty)$ and let $\sigma(x^1, \ldots, x^{k+1})$ be an arbitrary k-simplex of the triangulation containing x. Then there exist unique nonnegative numbers $\lambda_1, \ldots, \lambda_{k+1}$ summing up to 1 such that $x = \sum_{i=1}^{k+1} \lambda_i x^i$. Let $x^i = (p^i, t_i)$, then for all i, t_i is either equal to m or $m + 1$ for some m, $m = 1, 2, \ldots$,

$$p = \sum_{i=1}^{k+1} \lambda_i p^i \quad \text{and} \quad t = \sum_{i=1}^{k+1} \lambda_i t_i$$

with $m \leqslant t \leqslant m + 1$. The function g from $S \times [1, \infty)$ to R^{N+n} is defined by

$$g(x) = \sum_{i=1}^{k+1} \lambda_i h(x^i) = \sum_{i=1}^{k+1} \lambda_i z(p^i). \tag{2.2}$$

Clearly, g is a piecewise linear function and is independent of the choices of the simplices in which the argument lies. The continuous deformation algorithm to be described in the rest of this paper will generate points $x = (p, t)$ in $S \times [1, \infty)$ such that if $t = 1$

$$\begin{aligned} \bar{z}_{j,k}(p) &= \max_{(j,h) \in I(j)} \bar{z}_{j,h}(p) \quad \text{if } p_{j,k}/v_{j,k} > \min_{(i,h) \in I} p_{i,h}/v_{i,h} \quad \text{and} \\ \bar{z}_{j,k}(p) &\leqslant \max_{(j,h) \in I(j)} \bar{z}_{j,h}(p) \quad \text{if } p_{j,k}/v_{j,k} = \min_{(i,h) \in I} p_{i,h}/v_{i,h} \end{aligned} \tag{2.3}$$

and if $t > 1$

$$\begin{aligned} g_{j,k}(x) &= \max_{(j,h) \in I(j)} g_{j,h}(x) \quad \text{if } x_{j,k} > 0 \quad \text{and} \\ g_{j,k}(x) &\leqslant \max_{(j,h) \in I(j)} g_{j,h}(x) \quad \text{if } x_{j,k} = 0 \end{aligned} \tag{2.4}$$

where $\bar{z}$ is the piecewise linear approximation of z with respect to the V-triangulation of S on level 1 with grid size d_1^{-1} and where v is an arbitrary (interior) point of S being an initial guess of the solution to the NLCP. In fact, the algorithm is initiated at

$(v, 1)$. As will be shown in §6 the points $x = (p, t)$ in $S \times [1, \infty)$ satisfying (2.3) or (2.4) form, under a nondegeneracy assumption, piecewise linear (p.l.) paths of points in $S \times [1, \infty)$. Each such p.l. path is either a loop and consists of a finite number of pieces or is a route and consists of an infinite number of pieces. Exactly one route has the point $(v, 1)$ as an end point. This route exceeds each level m in a finite number of pieces when the route is followed by starting in $(v, 1)$. The route leading from $(v, 1)$ is generated by the algorithm and is followed by alternating linear programming steps in a system of $N + n + 1$ linear equations induced by (2.3) or (2.4) and replacement steps in the triangulation of $S \times [1, \infty)$, the latter yielding a movement from one simplex of the triangulation to an adjacent one and the first yielding a movement to follow a piece of the p.l. path in a given simplex. These steps are also described in §6.

In particular the points on $S \times \{1\}$ satisfying (2.3) form piecewise linear paths on level 1. Exactly one of these paths in $S \times \{1\}$ connects the point $(v, 1)$ with a point $(p^1, 1)$ satisfying (2.4) for $t = 1$ whereas all other paths not being a loop connect two points also satisfying (2.4) for $t = 1$. Moreover each end point of a p.l. path of points satisfying (2.3) is an end point of a p.l. path of points in $S \times [1, \infty)$ satisfying (2.4) unless $(v, 1)$ is the end point. In this way the p.l. paths of points in $S \times [1, \infty)$ satisfying (2.3) or (2.4) will be linked together in §6, yielding one (infinite) p.l. path having $(v, 1)$ as end point and being the path followed by the algorithm. So, starting in $(v, 1)$, the algorithm first generates a piecewise linear path in $S \times \{1\}$ whose points satisfy (2.3). When an end point $(p^1, 1)$ in $S \times \{1\}$ is found the algorithm continues in $S \times (1, \infty)$ by following the p.l. path of points satisfying (2.4) initiated in $(p^1, 1)$. As soon as an end point, say $(p^2, 1)$, on level 1 is reached the algorithm continues in $S \times \{1\}$ with the p.l. path of points satisfying (2.3) having $(p^2, 1)$ as one end point until the other end point, $(p^3, 1)$, of this p.l. path is reached. Then the algorithm continues again with a p.l. path in $S \times [1, \infty)$ satisfying (2.4) having $(p^3, 1)$ as end point, etc. So, each time the algorithm returns to $S \times \{1\}$, it continues with a p.l. path of points in $S \times \{1\}$ satisfying (2.3), leading to the other end point of that path. This end point is then also the end point of a p.l. path of points in $S \times [1, \infty)$ satisfying (2.4). Of course we must show that an end point of a path in $S \times \{1\}$ is an end point of a path of points satisfying (2.4) for $t = 1$.

In §3 we describe how a p.l. path of points $x = (p, 1)$ satisfying (2.3) can be followed by alternating pivot steps in a system of linear equations and replacement steps in the underlying V-triangulation of $S \times \{1\}$ having grid size d_1^{-1}. In fact, these steps are the steps of the product-ray algorithm of Doup and Talman [1]. This algorithm was developed to generate, for a sequence of increasing numbers d_m, points p^m satisfying (2.4) with $\bar{z}^m(p)$ instead of $g(x)$ where $\bar{z}^m$ is the p.l. approximation of z with respect to the V-triangulation with grid size d_m^{-1} and starting point p^{m-1} $(p^0 = v)$, $m = 1, 2, \ldots$. When the accuracy of approximation of p^m is not good enough the algorithm is restarted in p^m with grid size d_{m+1}^{-1} to find a hopefully better solution p^{m+1}, etc. In this paper we need the steps of this algorithm to follow the p.l. path of points on level 1 satisfying (2.3) leading from $(v, 1)$ to a point $(p^1, 1)$ satisfying (2.4) and to follow, also on level 1, a p.l. path of points satisfying (2.3) connecting two points $(p^2, 1)$ and $(p^3, 1)$ both satisfying (2.4). Since each level m is reached by the new algorithm in a finite number of steps, also any a priori given accuracy of approximation can be reached within a finite number of iterations. More precisely, let ϵ be an arbitrary positive number, let δ be so small that

$$\max_{(i,h)\in I} |p_{i,h} - q_{i,h}| < \delta \quad \text{implies} \quad \max_{(i,h)\in I} |z_{i,h}(p) - z_{i,h}(q)| < \epsilon$$

for all p and q in S, and let m_δ be such that the mesh of the V-triangulation of S (on level m) with grid size d_m^{-1} is smaller than δ for all $m \geqslant m_\delta$.

THEOREM 2.1. *Let ϵ, δ and m_δ be as defined above. Let $x = (p, t)$ satisfy* (2.4) *with* $t \geqslant m_\delta$. *Then, with* $\beta_j = \max\{z_{j,h}(p) | (j, h) \in I(j)\}$, $j \in I_N$, *we have*

$$-\epsilon < \beta_j < \epsilon \qquad j \in I_N,$$

$$\beta_j - \epsilon < z_{j,k}(p) < \beta_j + \epsilon \qquad \text{if } p_{j,k} > 0 \quad \text{and}$$

$$z_{j,k}(p) < \beta_j + \epsilon \qquad \text{if } p_{j,k} = 0.$$

PROOF. Let $\sigma(x^1, \ldots, x^{k+1})$ be a k-simplex in $S(U) \times [m, m+1]$ containing $x = (p, t)$ where $U = \{(i, h) \in I | p_{i,h} = 0\}$ and where $m \geqslant m_\delta$ is such that $m \leqslant t < m + 1$. Then x^i is equal to (p^i, t_i) with t_i equal to m or $m + 1$, $i = 1, \ldots, k + 1$, and there exist nonnegative numbers $\lambda_1, \ldots, \lambda_{k+1}$ summing up to one such that $x = \sum_i \lambda_i x^i$, i.e. $p = \sum_i \lambda_i p^i$ and $t = \sum_i \lambda_i t_i$. Further, let τ be an n-simplex of the V-triangulation of S (on level m) with grid size d_m^{-1} such that σ is contained in $\tau \times [m, m+1]$. Clearly, p and all the p^i's lie in τ and the diameter of τ is less than δ since $m \geqslant m_\delta$. Therefore, for all $j \in I_N$

$$|\beta_j| = |p_j^{\mathrm{T}} g_j(x)| = \left| p_j^{\mathrm{T}}(g_j(x) - h_j(x)) \right|$$

$$= \left| p_j^{\mathrm{T}} \left(\sum_{i=1}^{k+1} \lambda_i [g_j(x^i) - h_j(x)] \right) \right|$$

$$= \left| p_j^{\mathrm{T}} \left(\sum_{i=1}^{k+1} \lambda_i [z_j(p^i) - z_j(p)] \right) \right| < \epsilon.$$

Recall from (2.4) that $\beta_j = \max_h g_{j,h}(x) = g_{j,k}(x)$ for all (j, k) for which $p_{j,k} > 0$ and from (2.1) that $p_j^{\mathrm{T}} h_j(x) = 0$, for all $j \in I_N$. Furthermore, for all $(j, h) \in I(j)$, $j \in I_N$, we have that

$$|z_{j,h}(p) - \beta_j| = |z_{j,h}(p) - g_{j,h}(x)|$$

$$= \left| \sum_{i=1}^{k+1} \lambda_i [z_{j,h}(p) - z_{j,h}(p^i)] \right| < \epsilon$$

if $p_{j,h} > 0$ since then $\beta_j = g_{j,h}(x)$, and that according to (2.4)

$$z_{j,h}(p) - \beta_j \leqslant z_{j,h}(p) - g_{j,h}(x)$$

$$= \sum_i \lambda_i [z_{j,h}(p) - z_{j,h}(p^i)] < \epsilon$$

if $p_{j,h} = 0$, which completes the proof. ∎

The theorem implies that for any $\epsilon > 0$ there is a level m_ϵ such that if $x = (p, t)$ is generated by the algorithm with $t \geqslant m_\epsilon$, then $z_{j,h}(p) < \epsilon$ for all $(j, h) \in I$ and moreover $z_{j,h}(p) > -\epsilon$ for those (j, h) for which $p_{j,h} > 0$. We remark that only the product-ray algorithm seems to be a base of a continuous deformation algorithm on S. This is caused by the fact that only this algorithm yields an approximation having the same accuracy (ϵ) for each j. For a detailed discussion on this matter we refer to [1]. In that paper it is also argued that the V-triangulation (of S) is the only (programmable) triangulation of S which underlies this algorithm in a natural way. Furthermore, system (2.4) seems to be the natural way to follow a straight line of zero points of the function h.

3. The product-ray algorithm on S. We start by describing the V-triangulation of S (on level 1) with grid size d_1^{-1}, where d_1 is an arbitrary positive integer, and with starting point v. For simplicity we take v as an interior point of S (see [1] if v lies on the boundary of S).

Let T be a proper subset of the index set I such that $t(j) = |T_j| - 1 \geqslant 0$, $j \in I_N$, where T_j is defined by $T_j = T \cap I(j)$. With respect to (2.3) the set $A(T)$ is defined by

$$A(T) = \left\{ p \in S | p_{j,k}/v_{j,k} = \min_{(i,h)\in I} p_{i,h}/v_{i,h} \text{ if } (j,k) \notin T \right\}.$$

More precisely, $A(T)$ is the convex hull of the (starting) point v and the boundary face $S(I \setminus T)$ of S. Each $A(T)$ is subdivided in t-simplices as follows, where $t = |T| - N + 1 = \sum_{j=1}^N t(j) + 1$. For $j = 1, \dots, N$, let $\gamma_j = ((j, k_0^j), \dots, (j, k_{t(j)}^j))$ be a permutation of the $t(j) + 1$ elements in T_j, let Z_j^0 be given by $Z_j^0 = \{(j, k_0^j)\}$, $Z^0 = \bigcup_{j=1}^N Z_j^0$, $Z_j = T_j \setminus Z_j^0$, $Z = \bigcup_{j=1}^N Z_j$, and let $\gamma = (\gamma_1, \dots, \gamma_N)$.

DEFINITION 3.1. Let T be a proper subset of I and let the γ_j, Z_j^0, Z_j's and γ, Z^0, Z be as defined above, then the set $A(\gamma)$ is given by

$$A(\gamma) = \Bigg\{ x \in S | x = v + \alpha(Z^0) q^\gamma(Z^0) + \sum_{(j,k)\in Z} \alpha(j,k) q^\gamma(j,k),$$

$$0 \leqslant \alpha\left(j, k_{t(j)}^j\right) \leqslant \cdots \leqslant \alpha\left(j, k_1^j\right) \leqslant \alpha(Z^0) \leqslant 1,\ j \in I_N \Bigg\}$$

where the $(N+n)$-vector $q^\gamma(Z^0)$ is given by $q_j^\gamma(Z^0) = p_j(Z_j^0) - v_j$, $j \in I_N$, and where for $i = 1, \dots, t(j)$, $j \in I_N$, $q_h^\gamma(j, k_i^j) = 0$, $h \neq j$, and

$$q_j^\gamma(j, k_i^j) = p_j(\{(j, k_0^j), \dots, (j, k_i^j)\}) - p_j(\{(j, k_0^j), \dots, (j, k_{i-1}^j)\}),$$

where for $j \in I_N$ the $(n_j + 1)$-vector $p_j(K_j)$, $K_j \subset I(j)$, $K_j \neq \varnothing$, is given by

$$p_{j,k}(K_j) = \begin{cases} v_{j,k}\left(\sum_{(j,h)\in K_j} v_{j,h} \right)^{-1}, & (j,k) \in K_j, \\ 0, & (j,k) \notin K_j. \end{cases}$$

The vector $p_j(K_j)$ is the relative projection of v_j on the boundary of S^{n_j}. The set $A(T)$ is the union of $A(\gamma)$ over all permutation vectors γ of T. The t-dimensional set $A(\gamma)$, $t = |T| - N + 1$, is triangulated by the collection $G(\gamma)$ of t-simplices $\tau(w^1, \omega)$ with vertices $w^1, \dots, w^{t+1}$, where

(i) $w^1 = v + a(Z^0) d_1^{-1} q^\gamma(Z^0) + \sum_{(j,k)\in Z} a(j,k) d_1^{-1} q^\gamma(j,k)$, for nonnegative integers $a(Z^0)$ and $a(j,k)$, $(j,k) \in Z$, such that for all $j \in I_N$, $0 \leqslant a(j, k_{t(j)}^j) \leqslant \cdots \leqslant a(j, k_1^j) \leqslant a(Z^0) \leqslant d_1 - 1$;

(ii) $\omega = (\omega_1, \dots, \omega_t)$ is a permutation of the t elements consisting of Z^0 and the $t-1$ elements of Z such that for all $i = 1, \dots, t(j)$: $s > s'$ if $a(j, k_i^j) = a(j, k_{i-1}^j)$ where $\omega_s = (j, k_i^j)$, $\omega_{s'} = (j, k_{i-1}^j)$ if $i > 1$, and $\omega_{s'} = Z^0$ if $i = 1$;

(iii) $w^{i+1} = w^i + d_1^{-1} q^\gamma(\omega_i)$, $i = 1, \dots, t$, where $q^\gamma(Z^0)$ and $q^\gamma(j,k)$, $(j,k) \in Z$, are defined as before.

The union $G(T)$ of the $G(\gamma)$'s over all permutation vectors γ of T is a triangulation of $A(T)$ and the union G of the $G(T)$'s over all feasible T induces the V-triangulation of S with grid size d_1^{-1}.

In order to follow the p.l. path from v to an approximate solution, the product-ray algorithm on S generates for varying T a sequence of adjacent t-dimensional simplices in $A(T)$ having so-called T-complete common facets.

DEFINITION 3.2. Let T be a subset of I with $|T_j| \geqslant 1$, $j \in I_N$. For $k = t-1, t$, where $t = |T| - N + 1$, a k-simplex $\tau(w^1, \ldots, w^{k+1})$ is T-complete if the system of linear equations

$$\sum_{i=1}^{k+1} \lambda_i \begin{pmatrix} z(w^i) \\ 1 \end{pmatrix} + \sum_{(i,h) \notin T} \mu_{i,h} \begin{pmatrix} e(i,h) \\ 0 \end{pmatrix} - \sum_{j=1}^{N} \beta_j \begin{pmatrix} \bar{e}(j) \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}, \tag{3.1}$$

where $e(i,h)$ denotes the $(\sum_{j=1}^{i-1}(n_j+1)+h)$th unit vector in R^{N+n}, $\bar{e}(j) = \sum_{h=1}^{n_j+1} e(j,h)$, $j \in I_N$, and $\mathbf{0}$ is the $(N+n)$-dimensional zero vector in R^{N+n}, has a solution $\lambda_i^* \geqslant 0$, $i = 1, \ldots, k+1$, $\mu_{i,h}^* \geqslant 0$, $(i,h) \notin T$, and β_j^*, $j \in I_N$.

A solution will be denoted by $(\lambda^*, \mu^*, \beta^*)$. For a T-complete k-simplex with $k = t-1$ we assume that the system (3.1) has a unique solution $(\lambda^*, \mu^*, \beta^*)$, $\lambda_i^* > 0$, $i = 1, \ldots, t$ and $\mu_{i,h}^* > 0$, $(i,h) \notin T$, and that for $k = t$ the system has a line segment of solutions with at most one variable of (λ^*, μ^*) equal to zero (Nondegeneracy assumption). Therefore each T-complete $(t-1)$-simplex in $A(T)$ is a facet of either two T-complete simplices in $A(T)$ or of one in which case it lies on the boundary of $A(T)$. If τ lies in $A(T)$ and $p = \sum_i \lambda_i^* w^i$, then according to (3.1) the point p satisfies (2.3).

DEFINITION 3.3. A T-complete $(t-1)$-simplex $\tau(w^1, \ldots, w^t)$ in $A(T)$ is complete if for all x in τ: $x_{i,h} = 0$, $(i,h) \notin T$.

Observe that we allow T to be equal to I. If $p = \sum_i \lambda_i^* w^i$ lies in a complete simplex, then it satisfies (2.4).

As described in Doup and Talman [1] the T-complete t-simplices in $A(T)$, $T \subset I$, determine sequences of adjacent simplices of varying dimension such that each path is either a loop or has two end points. Exactly one end point is the zero-dimensional simplex $\tau(v)$, whereas all other end points are complete simplices. Exactly one sequence determines therefore a p.l. path of points satisfying (2.3) which connects v with a point satisfying (2.4) whereas all other sequences determine p.l. paths which connect two points satisfying (2.4). We will now give the replacement steps occurring in the algorithm which follows such a path.

Let $\tau(w^1, \omega)$ be a T-complete t-simplex in $G(\gamma)$ such that the T-complete facet of τ opposite vertex w^s, $1 \leqslant s \leqslant t+1$, is a facet of another T-simplex $\bar{\tau}(\bar{w}^1, \bar{\omega})$ in $G(\gamma)$, then $\bar{\tau}$ can be obtained from τ as given in Table 1, where $e(Z^0) = \sum_{j=1}^{N} e(j, k_0^j)$, $a(j, k_0^j) = a(Z^0)$, $a_{i,h} = a(i,h)$, $(i,h) \in T$, and $a_{i,h} = 0$ if $(i,h) \notin T$. The vertex w^s is replaced by the new vertex of $\bar{\tau}$. Now consider the case that the T-complete facet of τ opposite vertex w^s, $1 \leqslant s \leqslant t+1$, is not a facet of another t-simplex in $G(\gamma)$.

TABLE 1

s is the Index of the Vertex of τ to Be Replaced

	$\bar{w}^1$	$\bar{\omega}$	$\bar{a}$
$s = 1$	$w^1 + d_1^{-1} q^{\gamma}(\omega_1)$	$(\omega_2, \ldots, \omega_t, \omega_1)$	$a + e(\omega_1)$
$1 < s < t+1$	w^1	$(\omega_1, \ldots, \omega_{s-2}, \omega_s, \omega_{s-1}, \ldots, \omega_t)$	a
$s = t+1$	$w^1 - d_1^{-1} q^{\gamma}(\omega_t)$	$(\omega_t, \omega_1, \ldots, \omega_{t-1})$	$a - e(\omega_t)$

LEMMA 3.4. *Let $\tau(w^1, \omega)$ be a T-complete t-simplex in $G(\gamma)$. The T-complete facet of τ opposite w^s, $1 \leqslant s \leqslant t+1$, lies on the boundary of $A(\gamma)$ iff*
(a) $s = 1$, $\omega_1 = Z^0$ *and* $a(Z^0) = d_1 - 1$;
(b) $1 < s < t+1$, $\omega_s = (j, k_i^j)$ *for certain* $j \in I_N$, $1 \leqslant i \leqslant t(j)$, $\omega_{s-1} = (j, k_{i-1}^j)$ *if* $i > 1$ *and* $\omega_{s-1} = Z^0$ *if* $i = 1$, *and* $a(\omega_{s-1}) = a(\omega_s)$;
(c) $s = t+1$, $\omega_t = (j, k_{t(j)}^j)$ *for certain* $j \in I_N$ *and* $a(\omega_t) = 0$.

In case (a) the $(t-1)$-facet of τ opposite vertex w^1 is complete. In case (b) the facet of τ opposite vertex w^s is a facet of the T-complete t-simplex $\bar{\tau}(\bar{w}^1, \bar{\omega})$ in $G(\bar{\gamma})$ where

$$\bar{\gamma}_h = \begin{cases} \gamma_h, & h \neq j, \\ \left(\left(j, k_0^j\right), \ldots, \left(j, k_{i-2}^j\right), \left(j, k_i^j\right), \left(j, k_{i-1}^j\right), \ldots, \left(j, k_{t(j)}^j\right)\right), & h = j, \end{cases}$$

and $\bar{w}^1 = w^1$, $\bar{a} = a$ and

$$\bar{\omega} = \begin{cases} \left(\omega_1, \ldots, \omega_{s-2}, \omega_s, \omega_{s-1}, \omega_{s+1}, \ldots, \omega_t\right), & i > 1, \\ \left(\omega_1, \ldots, \omega_{s-2}, \bar{Z}^0, \left(j, k_0^j\right), \omega_{s+1}, \ldots, \omega_t\right), & i = 1, \end{cases}$$

with $\bar{Z}_j^0 = \{(j, k_1^j)\}$ and $\bar{Z}_h^0 = \{(h, k_0^h)\}$, $h \neq j$, if $i = 1$.

In case (c) τ is the T-complete $(t-1)$-simplex $\bar{\tau}(\bar{w}^1, \bar{\omega})$ with $\bar{w}^1 = w^1$, $\bar{\omega} = (\omega_1, \ldots, \omega_{t-1})$ and $\bar{a} = a$, and lies in $G(\bar{\gamma})$ where $\bar{\gamma}$ is given by

$$\bar{\gamma}_h = \begin{cases} \left(\left(j, k_0^j\right), \ldots, \left(j, k_{t(j)-1}^j\right)\right), & h = j, \\ \gamma_h, & h \neq j. \end{cases}$$

Furthermore we have the following lemma if a t-simplex $\tau(w^1, \omega)$ in $G(\gamma)$ is $T \cup \{(j, k)\}$-complete but not complete.

LEMMA 3.5. *Let $\tau(w^1, \omega)$ be a $T \cup \{(j, k)\}$-complete t-simplex in $G(\gamma)$, for some $(j, k) \in I \setminus T$, with γ a permutation vector of T, and $t < n$. Then τ is a facet of the $(t+1)$-simplex $\bar{\tau}(\bar{w}^1, \bar{\omega})$, with $\bar{w}^1 = w^1$, $\bar{\omega} = (\omega_1, \ldots, \omega_t, (j, k))$ and $\bar{a} = a$, in $G(\bar{\gamma})$ where $\bar{\gamma}$ is given by*

$$\bar{\gamma}_h = \begin{cases} \left(\left(j, k_0^j\right), \ldots, \left(j, k_{t(j)}^j\right), (j, k)\right), & h = j, \\ \gamma_h, & h \neq j. \end{cases}$$

We will now give the steps of the algorithm, omitting the initialization step, in order to generate either a path of adjacent simplices from the 0-dimensional simplex $\tau(v)$ to a complete simplex, say τ^0, or to generate such a path from a complete simplex τ^1, $\tau^1 \neq \tau^0$, to another complete simplex τ^2. The number $\bar{s}$ is the index of the vertex of τ for which $z(w^{\bar{s}})$ has to be calculated.

Step 1. Calculate $z(w^{\bar{s}})$. Perform a pivot step by bringing $(z^{\mathrm{T}}(w^{\bar{s}}), 1)^{\mathrm{T}}$ in the linear system

$$\sum_{i=1, i \neq s}^{t+1} \lambda_i \begin{pmatrix} z(w^i) \\ 1 \end{pmatrix} + \sum_{(i,h) \notin T} \mu_{i,h} \begin{pmatrix} e(i, h) \\ 0 \end{pmatrix} - \sum_{j=1}^{N} \beta_j \begin{pmatrix} \bar{e}(j) \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}.$$

Either $\mu_{i,h}$ becomes zero for some $(i, h) \notin T$ then go to Step 3, or λ_s becomes zero for exactly one vertex $w^s \neq w^{\bar{s}}$. The facet opposite vertex w^s is T-complete.

Step 2. If $s = 1$, $\omega_1 = Z^0$ and $a(Z^0) = d_1 - 1$ then the facet of τ opposite vertex w^1 is a complete $(t-1)$-simplex and the algorithm terminates.

If $1 < s < t+1$, $\omega_s = (j, k_i^j)$ for certain $j \in I_N$, $1 \leqslant i \leqslant t(j)$, $\omega_{s-1} = (j, k_{i-1}^j)$ if $i > 1$, $\omega_{s-1} = Z^0$ if $i = 1$, and if $a(\omega_{s-1}) = a(\omega_s)$ then τ and γ are adapted according to Lemma 3.4(b).

If $s = t+1$, $\omega_t = (j, k_{t(j)}^j)$ for certain $j \in I_N$ and $a(\omega_t) = 0$, then the dimension is decreased; set $t = t-1$, $T = T \setminus \{(j, k_{t(j)}^j)\}$ and $p = (j, k_{t(j)}^j)$ while τ and γ are adapted according to Lemma 3.4(c) and go to Step 4.

In all other cases $\tau(w^1, \omega)$ and a are adapted according to Table 1.

Return to Step 1 with $\bar{s}$ the index of the new vertex of τ.

Step 3. If $t = n$, then $\tau(w^1, \omega)$ is a complete n-simplex and the algorithm terminates; otherwise $\tau(w^1, \omega)$ and γ are adapted according to Lemma 3.5, set $t = t+1$ and $T = T \cup \{(i, h)\}$. Return to Step 1 with $\bar{s}$ the index of the new vertex of τ.

Step 4. Perform a pivot step by bringing $(e^{\mathrm{T}}(p), 0)^{\mathrm{T}}$ in the linear system

$$\sum_{i=1}^{t+1} \lambda_i \begin{pmatrix} z(w^i) \\ 1 \end{pmatrix} \sum_{\substack{(i,h) \notin T \\ (i,h) \neq p}} \mu_{i,h} \begin{pmatrix} e(i,h) \\ 0 \end{pmatrix} - \sum_{j=1}^{N} \beta_j \begin{pmatrix} \bar{e}(j) \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}.$$

If for some $(i, h) \notin T$, $\mu_{i,h}$ becomes zero then go to Step 3, otherwise return to Step 2 with s the index for which λ_s becomes zero.

We can distinguish the following three initializations of the algorithm:

(1) With the 0-dimensional simplex $\tau(v)$;

(2) for some γ with a complete T-complete $(t-1)$-simplex $\tilde{\tau}(\tilde{w}^1, \tilde{\omega})$ in $A(\gamma) \cap S(I \setminus T)$, where $\tilde{\omega}$ is a permutation of the elements in Z, with basic solution $\lambda_i^* > 0$, $i = 1, \ldots, t$, $\mu_{i,h}^* > 0$, $(i, h) \notin T$, and β_j^*, $j \in I_N$;

(3) for some γ with a complete I-complete n-simplex $\tilde{\tau}(\tilde{w}^1, \tilde{\omega})$ in $A(\gamma)$, where $\tilde{\omega}$ is a permutation of Z^0 and the elements of Z, with basic solution $\lambda_i^* > 0$, $i = 1, \ldots, n+1$, and β_j^*, $j \in I_N$.

In the first case the algorithm is initialized with the Z^0-complete 1-simplex $\tau(w^1, \omega)$, $Z^0 = \bigcup_{j=1}^N \{(j, k_0^j)\}$ where the index (j, k_0^j) is such that $z_{j, k_0^j}(v) = \max_k z_{j,k}(v)$, $j \in I_N$, $w^1 = v$, $\omega = (Z^0)$, $a(i, h) = 0$, $(i, h) \in I$, and with basic solution $\lambda_1 = 1$, $\mu_{i,h} = z_{i, k_0^i}(v) - z_{i,h}(v)$, $(i, h) \notin Z^0$, and $\beta_j = z_{j, k_0^j}(v)$, $j \in I_N$. The index $\bar{s}$ is set equal to 2 and the algorithm starts with Step 1.

In the second case the algorithm is initialized with the T-complete t-simplex $\tau(w^1, \omega)$ with $w^1 = \tilde{w}^1 - d_1^{-1} q^{\gamma}(Z^0)$, $\omega = (Z^0, \tilde{\omega}_1, \ldots, \tilde{\omega}_{t-1})$, $a = \tilde{a} - e(Z^0)$ where $\tilde{a}$ induces $\tilde{w}^1$, and the basic solution $\lambda_i = \lambda_{i-1}^*$, $i = 2, \ldots, t+1$, $\mu_{i,h} = \mu_{i,h}^*$, $(i, h) \notin T$ and $\beta_j = \beta_j^*$, $j \in I_N$. The index $\bar{s}$ is set equal to 1 and the algorithm starts with Step 1.

In the third case the algorithm is initialized with the I-complete n-simplex $\tau(w^1, \omega)$ with $w^1 = \tilde{w}^1$, $\omega = \tilde{\omega}$ and $a = \tilde{a}$. Let (i, h) be the unique index not in $Z^0 \cup Z$ then we set $T = I \setminus \{(i, h)\}$ and $p = (i, h)$ while the basic solution of τ is given by $\lambda_i = \lambda_i^*$, $i = 1, \ldots, n+1$, $\beta_j = \beta_j^*$, $j \in I_N$. Now the algorithm starts with Step 4.

This completes the description of the steps of the simplicial variable dimension (restart) algorithm on S to generate a sequence of adjacent simplices in $A(T)$ having common T-complete facets, for varying T in order to trace a piecewise linear path of points satisfying (2.3). If not a loop such a path either connects the point v with a point p^1 in a complete simplex satisfying (2.4) or it connects two points in complete simplices both satisfying (2.4).

4. Triangulation of $S \times [1, \infty)$. In this section we describe the triangulation of $S \times [1, \infty)$ which underlies the continuous deformation algorithm on S. As said before the triangulation of $S \times [1, \infty)$ is such that on level m, $m = 1, 2, \ldots,$ $S \times \{m\}$ is triangulated according to the V-triangulation with grid size d_m^{-1} and starting point v, where for an arbitrary sequence $k_1, k_2, \ldots,$ of integers larger than one $d_m = k_{m-1} d_{m-1}$. The point v is the initial solution guess of the NLCP with which the algorithm starts on level 1. When there is no a priori information about the location of a solution, v can be chosen equal to the barycenter of S. The number d_1^{-1} is the grid size of the V-triangulation on level 1 and might be chosen equal to one if there is no information about the location of a solution. The V-triangulation of S on level m is denoted by V_m. The triangulation of $S \times [1, \infty)$ is in fact such that for each m, $m = 1, 2, \ldots,$ the set $S \times [m, m+1]$ is triangulated into $(n+1)$-simplices with only vertices on either the level m or on the level $m+1$. More precisely, for any n-simplex τ^m of the V-triangulation of S on level m with grid size d_m^{-1} the cylinder $\tau^m \times [m, m+1]$ is triangulated in $(n+1)$-simplices similar as described in [12] and [13] for the continuous deformation algorithm with arbitrary grid refinement on the n-dimensional unit simplex S^n. Notice that on level $m+1$ the grid is refined with a factor of k_m so that $\tau^m \times \{m+1\}$ consists of n to the power k_m n-dimensional simplices. Each such simplex will be connected with one of the $n+1$ vertices of τ^m on level m in such a way that a triangulation of $\tau^m \times [m, m+1]$ is obtained. On the other hand the simplex τ^m on level m is connected with exactly one vertex on level $m+1$. This vertex is called the centre point of $\tau^m \times \{m+1\}$. To describe the triangulation of $S \times [m, m+1]$ in full detail we have to redefine the full-dimensional sets $A(\gamma)$ as follows. Let $\gamma_j = ((j, k_0^j), \ldots, (j, k_{t(j)}^j))$, $t(j) = n_j$, be a permutation of the $n_j + 1$ elements of $I(j)$, $Z_j^0 = \{(j, k_0^j)\}$, $Z_j = I(j) \setminus Z_j^0$, $j \in I_N$, and let $\gamma = (\gamma_1, \ldots, \gamma_N)$, $Z^0 = \bigcup_{j=1}^N Z_j^0$ and $Z = \bigcup_{j=1}^N Z_j$. Then for $j_0 \in I_N$ we define

$$A(\gamma, j_0) = \Big\{ x \in S | x = v + \alpha(Z^0) q^\gamma(Z^0) + \sum_{(j,k) \in Z} \alpha(j,k) q^\gamma(j,k), \text{ where}$$

$$0 \leqslant \alpha\big(j, k_{t(j)}^j\big) \leqslant \cdots \leqslant \alpha\big(j, k_1^j\big) \leqslant \alpha(Z^0) \leqslant 1,\ j \in I_N, \text{ and}$$

$$\alpha\big(j_0, k_{t(j_0)}^{j_0}\big) = 0 \Big\}.$$

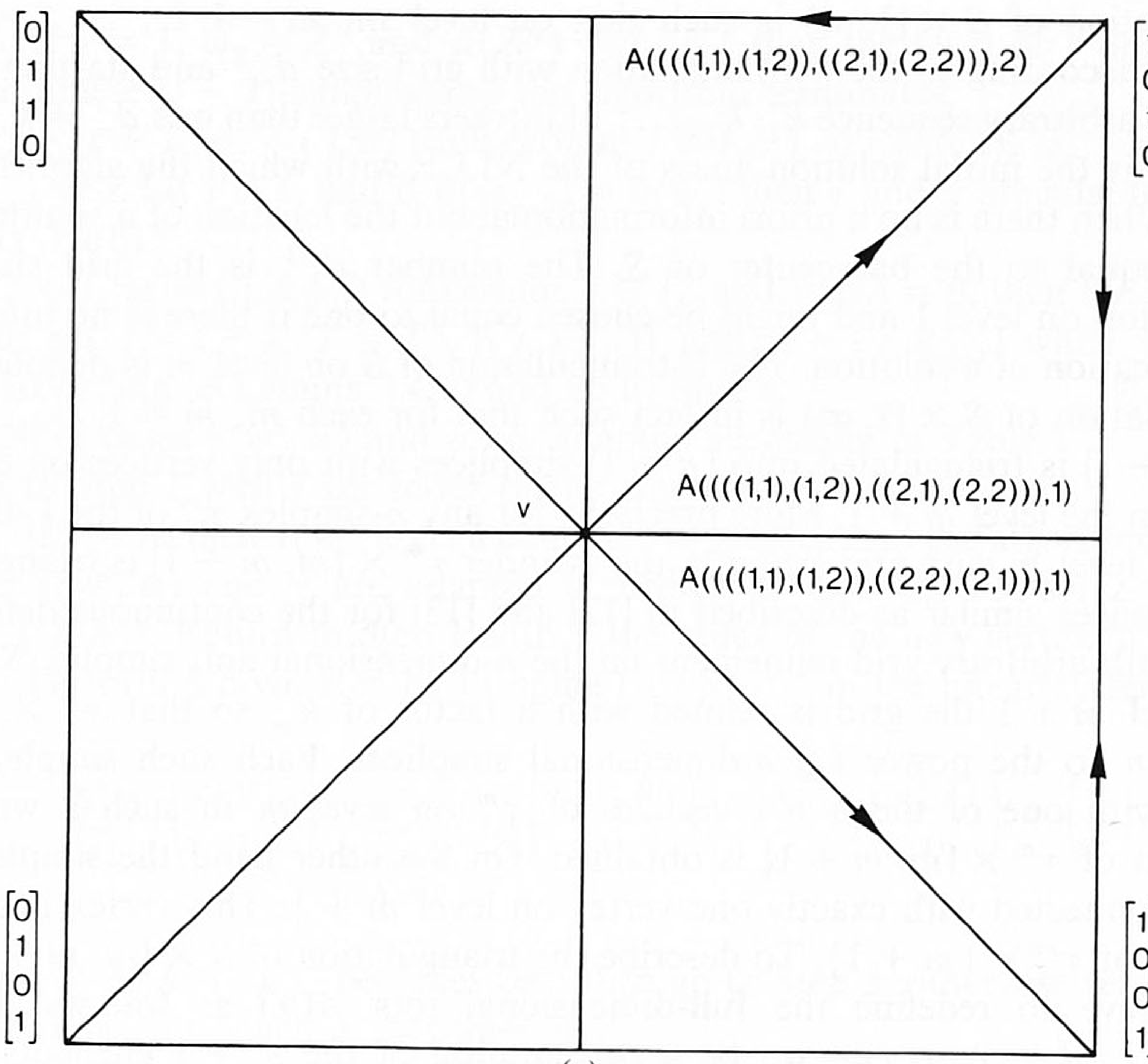
In fact $A(\gamma, j_0)$ is equal to $A(\bar{\gamma})$, with $\bar{\gamma}_h = \gamma_h$, $h \neq j_0$, and $\bar{\gamma}_{j_0} = ((j_0, k_0^{j_0}), \ldots, (j_0, k_{t(j_0)-1}^{j_0}))$. The set $G_m(\gamma, j_0)$ is the collection of n-simplices $\tau(w^1, \omega)$ with vertices $w^1, \ldots, w^{n+1}$ such that

(i) $w^1 = v + a(Z^0) d_m^{-1} q^\gamma(Z^0) + \Sigma_{(j,k) \in Z} a(j,k) d_m^{-1} q^\gamma(j,k)$ for nonnegative integers $a(Z^0)$ and $a(j,k)$, $(j,k) \in Z$, such that $a(j_0, k_{t(j_0)}^{j_0}) = 0$ and $0 \leqslant a(j, k_{t(j)}^j) \leqslant \cdots \leqslant a(Z^0) \leqslant d_m - 1$, $j \in I_N$;

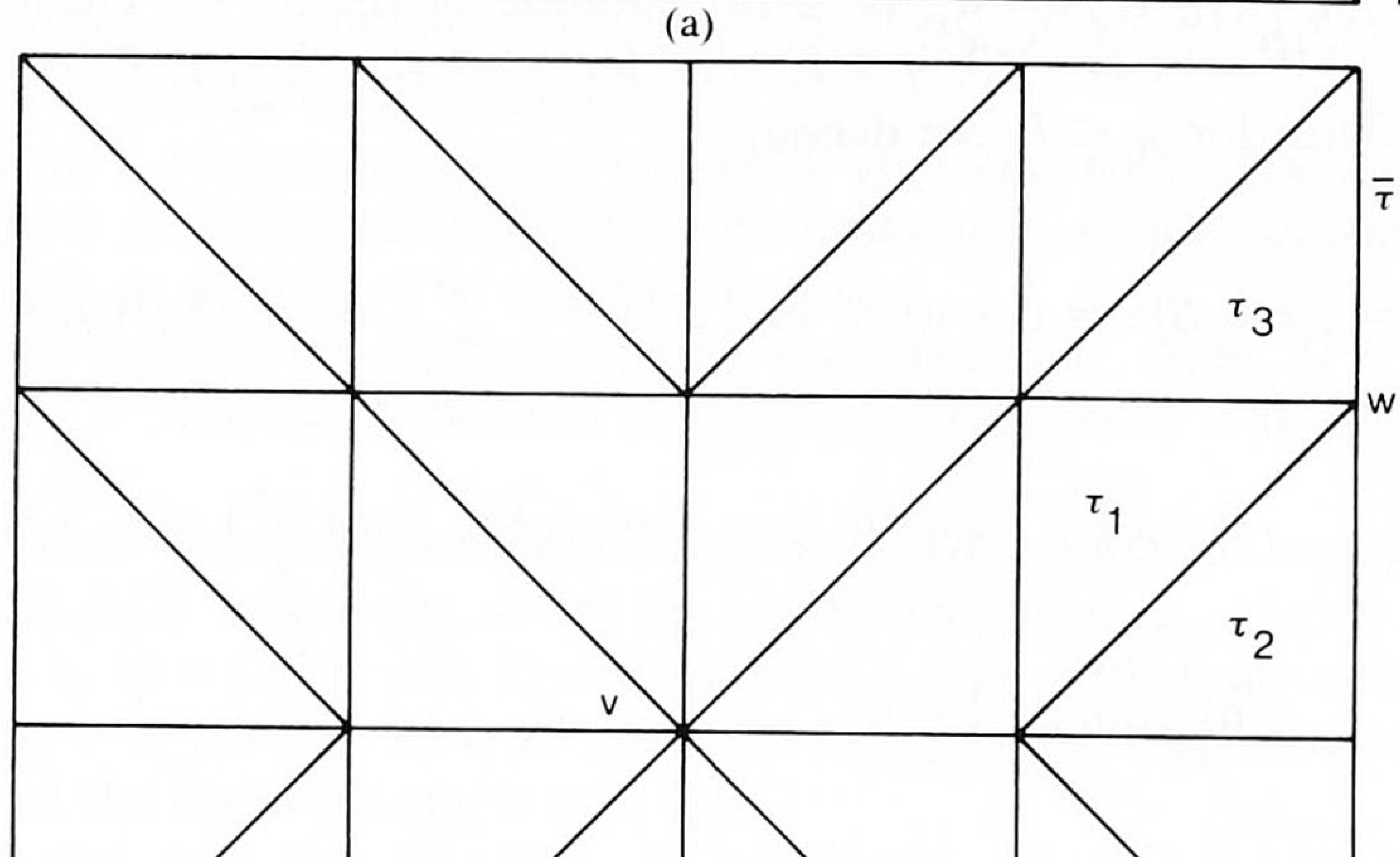
(ii) $\omega = (\omega_1, \ldots, \omega_{n+1})$ is a permutation of $n+1$ elements consisting of Z^0 and the n elements of Z such that $\omega_{n+1} = (j_0, k_{t(j_0)}^{j_0})$ and for all $i = 1, \ldots, t(j)$: $s > s'$ if $a(j, k_i^j) = a(j, k_{i-1}^j)$ where $\omega_s = (j, k_i^j)$ and $\omega_{s'} = (j, k_{i-1}^j)$ if $i > 1$ and $\omega_{s'} = Z^0$ if $i = 1$, $j \in I_N$;

(iii) $w^{i+1} = w^i + d_m^{-1} q^\gamma(\omega_i)$, $i = 1, \ldots, n+1$, with the convention $i + 1 = 1$ in the case $i = n+1$.

For $S = S^1 \times S^1$ the V-triangulation of S with grid size $1/2$, being the union of the $G_m(\gamma, j_0)$'s over all permutation vectors γ and indices j_0 is illustrated in Figure 1. The arrows on the edges of the simplices in Figure 1b give the order of the vertices in the simplices $\tau(w^1, \omega) = \tau(w^1, w^2, w^3)$. The simplices τ_1, τ_2 and τ_3 all lie in $A((((1,1),(1,2)),((2,1),(2,2))),1)$.



(a)



(b)

FIGURE 1. The V-Triangulation of $S = S^1 \times S^1$ with Grid Size 1/2.

Observe that $G_m(\gamma, j_0)$ is equivalent to $G(\bar{\gamma})$ with grid size d_m^{-1} so that the union of $G_m(\gamma, j_0)$ over all permutations γ of I and indices $j_0 \in I_N$ is the V-triangulation of S with grid size d_m^{-1}.

Now let s^m be a function from the grid points in $A(\gamma, j_0)$ to I_{n+1} such that each n-simplex $\tau(w^1, \omega)$ in $G_m(\gamma, j_0)$ is completely labelled, i.e., $\{s^m(w^i) | i = 1, \ldots, n+1\} = I_{n+1}$. The function s^m follows ideas of Todd [16] and Eaves and Saigal [5], see also

[12], and is utilized to make the union of the triangulations of each $\tau \times [m, m+1]$, $\tau \in G(\bar{\gamma})$, over all $\bar{\gamma}$, a triangulation of $S \times [m, m+1]$. In the sequel we will use the function s^m given by

$$s^m(x) = 1 + \left(a(Z^0) + \sum_{(i,h)\in Z} a(i,h)\right) \bmod (n+1),$$

where $x = v + a(Z^0)d_m^{-1}q^\gamma(Z^0) + \Sigma_{(i,h)\in Z}a(i,h)d_m^{-1}q^\gamma(i,h)$, is a grid point of V_m in $A(\gamma, j_0)$.

We are now ready to triangulate $S \times [m, m+1]$ for some given $m \geqslant 1$. First we choose nonnegative integers $\theta_1^m, \ldots, \theta_{n+1}^m$ with sum equal to the grid refinement factor $k_m = d_{m+1}/d_m$. For any n-simplex $\tau(w^1, \omega)$ of V_m we call the point $c(\tau) = \Sigma_{i=1}^{n+1}\delta_i k_m^{-1} w^i$, with $\delta_i = \theta^m_{s^m(w^i)}$, $i = 1, \ldots, n+1$, the centre point of τ. In §6 we discuss how the θ_i^m's should be chosen in the algorithm. Observe that $c(\tau)$ is a grid point of V_{m+1}. The triangulation of $S \times [m, m+1]$ is completely determined by the numbers $\theta_1^m, \ldots, \theta_{n+1}^m$.

To triangulate $\tau(w^1, \omega) \times [m, m+1]$, τ in V_m, we define for any proper subset T of $\{\omega_1, \ldots, \omega_{n+1}\}$ the regions $\mathring{A}(T, \tau)$ in τ by

$$\mathring{A}(T,\tau) = \left\{x \in \tau \mid x = c(\tau) + \sum_{j\in T} \alpha_j q^\gamma(j),\ \alpha_j > 0,\ j \in T\right\}.$$

Let $A(T, \tau)$ be the closure of $\mathring{A}(T, \tau)$, then on level $m+1$ $A(T, \tau)$ is triangulated by V_{m+1} in t-simplices $\sigma(y^1, \pi(T))$ with vertices $y^1, \ldots, y^{t+1}$ in τ such that

(1) $y^1 = c(\tau) + \Sigma_{h=1}^{n+1} R_{\omega_h} d_{m+1}^{-1} q^\gamma(\omega_h)$ with $R_{\omega_h} \geqslant 0$, $\omega_h \in T$, and $R_{\omega_h} = 0$, $\omega_h \notin T$;

(2) $\pi(T) = (\pi_1, \ldots, \pi_t)$ is a permutation of the t elements of T;

(3) $y^{i+1} = y^i + d_{m+1}^{-1} q^\gamma(\pi_i)$, $i = 1, \ldots, t+1$.

Notice that $y^1 = \Sigma_{h=1}^{n+1}(\delta_h - R_{\omega_h} + R_{\omega_{h-1}})k_m^{-1}w^h$ and that, for $i = 1, \ldots, t$, $y^{i+1} = y^i + k_m^{-1}(w^{s+1} - w^s)$ with s such that $\pi_i = \omega_s$.

In the sequel the $(N+n)$-vector R is defined by $R_{j,k} = R_{\omega_h}$ if $\omega_h = (j, k)$, $(j, k) \in Z$, $R_{j,k} = R_{Z^0}$ if $(j, k) \in Z^0$, and $R_{j,k} = 0$, otherwise. For $n = 2$ and $k_m = 4$, the sets $A(T, \tau)$ and the triangulation of these sets in t-simplices are illustrated in Figure 2 if $\delta_1 = 1$, $\delta_2 = 1$ and $\delta_3 = 2$ and in Figure 3 if $\delta_1 = 2$, $\delta_2 = 0$ and $\delta_3 = 2$. The arrows on the edges of the simplices give the order of the vertices in a simplex $\sigma(y^1, \pi(T))$. The simplices σ_1, σ_2 and σ_3 all lie in $A(\{1,3\}, \tau)$. Clearly, the boundary of $A(T, \tau)$ is given by

$$\operatorname{bd} A(T,\tau) = \bigcup_{\omega_s\in T} (A(T,\tau) \cap \tau_s) \cup \left(\bigcup_{\omega_s \in T} A(T\setminus\{\omega_s\}, \tau)\right)$$

where τ_s is the facet of τ opposite vertex w^s of τ. The next lemma describes when a facet of $\sigma(y^1, \pi(T))$ in $A(T, \tau)$ lies in bd $A(T, \tau)$.

LEMMA 4.1. *Let $\sigma(y^1, \pi(T))$ be a t-simplex in $A(T, \tau)$. Then the facet of σ opposite vertex y^p, for some $1 \leqslant p \leqslant t+1$, lies in the boundary of $A(T, \tau)$ iff one of the following cases holds*:

(1) $p = 1$, $\delta_s - R_{\omega_s} = 1$ *with* $\omega_s = \pi_1$, *and* $\omega_{s-1} \notin T$,

(2) $1 < p < t+1$, $\delta_s - R_{\omega_s} + R_{\omega_{s-1}} = 0$ *with* $\omega_s = \pi_p$, *and* $\omega_{s-1} = \pi_{p-1}$,

(3) $p = t+1$ *and* $R_{\omega_s} = 0$ *with* $\omega_s = \pi_t$.

In Figure 2b case (1) is illustrated by σ^1 and $y^1 = y$, case (2) is illustrated by σ^2 and $y^2 = y$ and case (3) is illustrated by σ^3 and $y^3 = y$. The same holds for Figure 3b. In

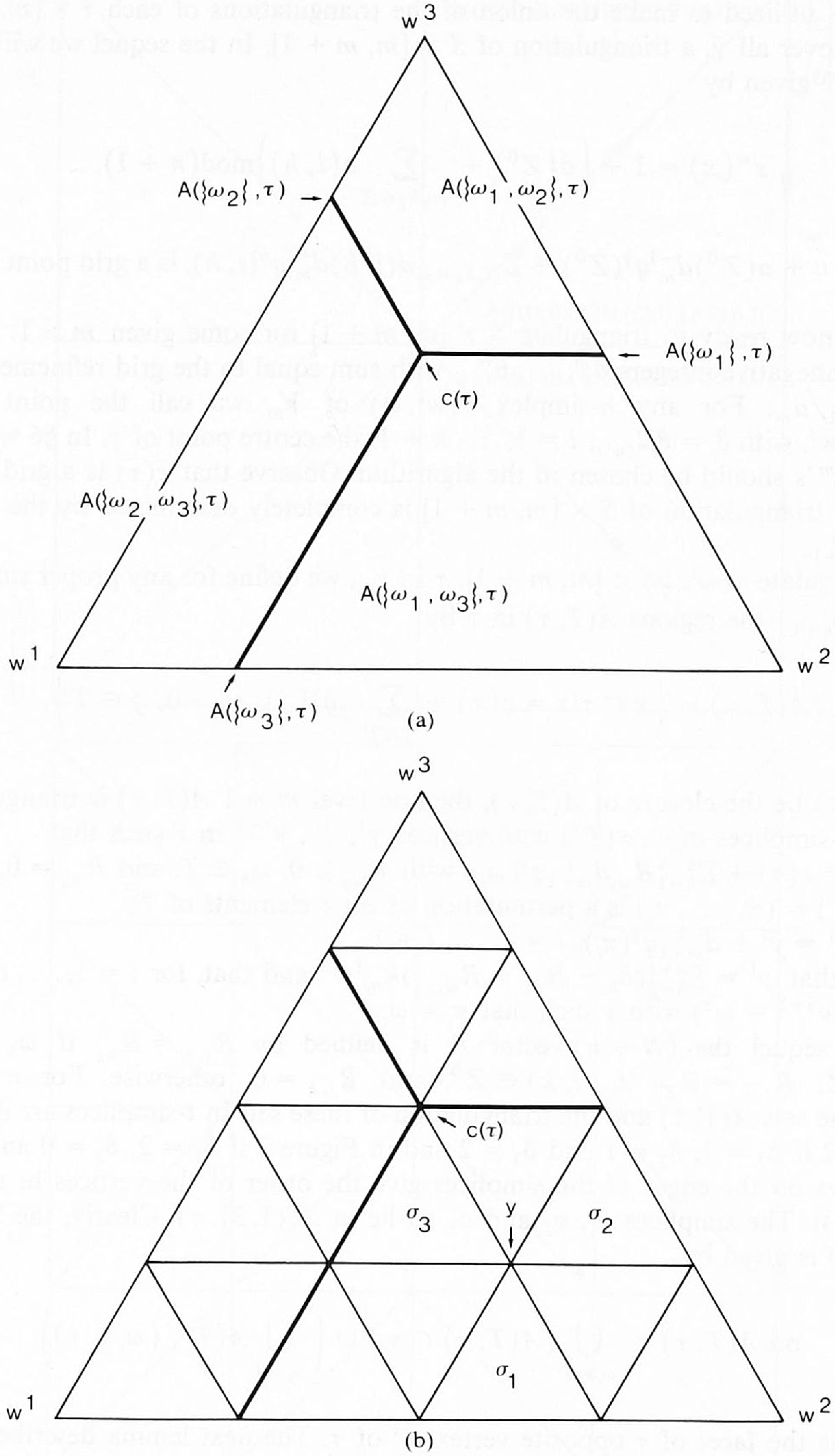


FIGURE 2. The Triangulation of τ in $\sigma(y^1, \pi(T))$'s on Level $m+1$, $\tau \in V_m$, When $\delta_1 = 1$, $\delta_2 = 1$, $\delta_3 = 2$, $k_m = 4$, $n = 2$.

the first two cases the facet of σ opposite vertex y^p for some p, $1 \leqslant p < t+1$, lies in $\mathrm{co}(\{w^1, \ldots, w^{s-1}, w^{s+1}, \ldots, w^{n+1}\})$ and in the third case the facet of σ opposite vertex y^{t+1} lies in $A(T \setminus \{\pi_t\}, \tau)$.

If the facet of σ opposite a vertex y^p for some p, $1 \leqslant p \leqslant t+1$, does not lie in the boundary of $A(T, \tau)$, then this facet is a facet of just one other t-simplex $\bar{\sigma}(\bar{y}^1, \bar{\pi}(T))$ in $A(T, \tau)$ where the parameters $\bar{y}^1$, $\bar{\pi}(T)$ and $\bar{R}$ are given in Table 2. Now

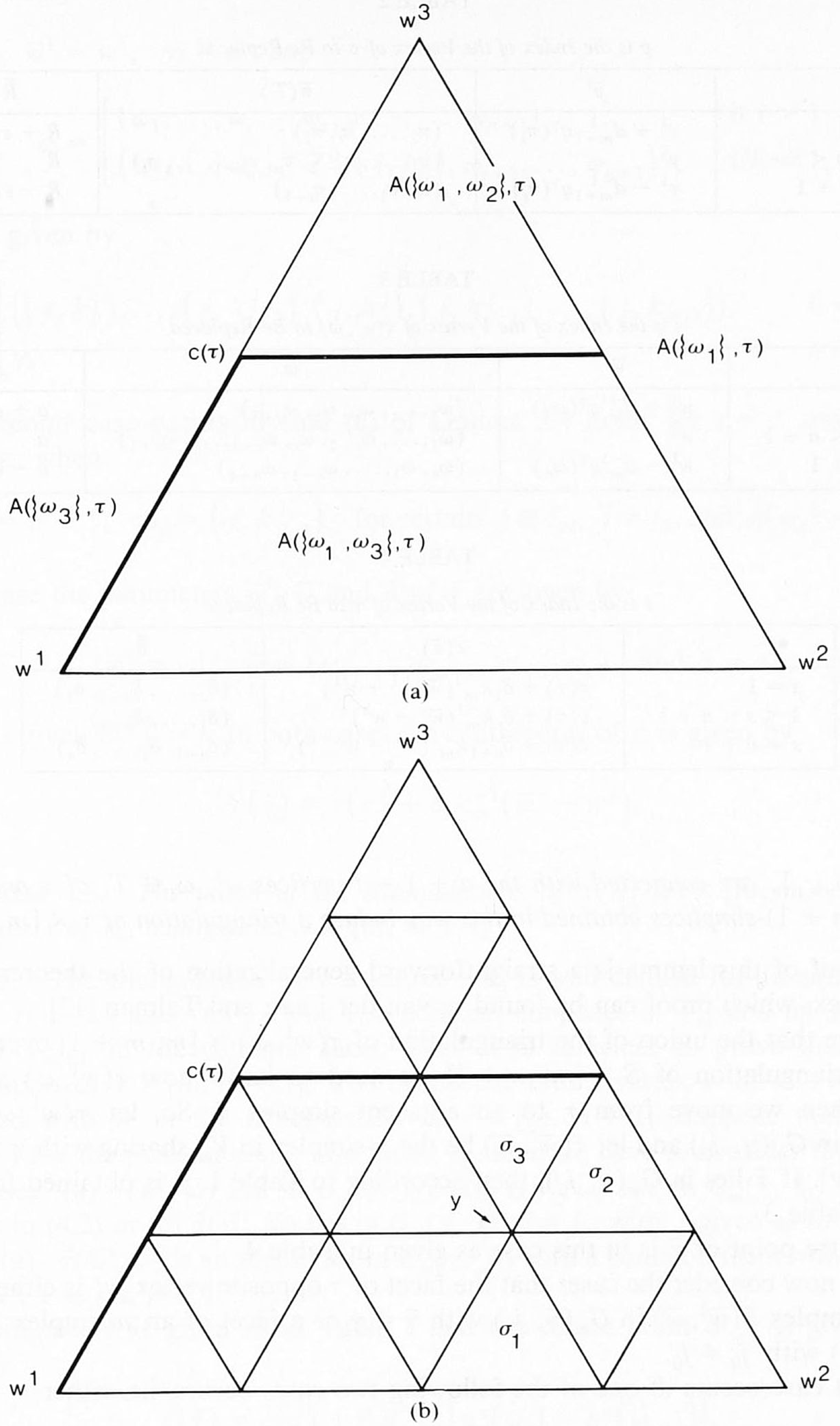


FIGURE 3. The Triangulation of τ in $\sigma(y^1, \pi(T))$'s on Level $m + 1$, $\tau \in V_m$, When $\delta_1 = 2$, $\delta_2 = 0$, $\delta_3 = 2$, $k_m = 4$, $n = 2$.

$\tau \times [m, m + 1]$ is triangulated by $(n + 1)$-simplices ψ^γ where for some $T \subset \{\omega_1, \dots, \omega_{n+1}\}$ and $\sigma(y^1, \pi(T))$ in $A(T, \tau)$

$$\psi^\gamma = \mathrm{co}\Big(\mathrm{co}\big(\{w^i | \omega_i \notin T\}\big) \times \{m\}, \sigma\big(y^1, \pi(T)\big) \times \{m + 1\}\Big).$$

LEMMA 4.2. *Let $\tau(w^1, \omega)$ be an n-simplex in $G_m(\gamma, j_0)$ with centre point $c(\tau)$. If the t-simplices $\sigma(y^1, \pi(T))$ of V_{m+1} in $A(T, \tau)$ on level $m + 1$, for proper subsets T of*

TABLE 2

p is the Index of the Vertex of σ to Be Replaced

	$\bar{y}^1$	$\bar{\pi}(T)$	$\bar{R}$
$p = 1$	$y^1 + d_{m+1}^{-1} q^\gamma(\pi_1)$	$(\pi_2, \ldots, \pi_t, \pi_1)$	$R + e(\pi_1)$
$1 < p < t + 1$	y^1	$(\pi_1, \ldots, \pi_{p-2}, \pi_p, \pi_{p-1}, \ldots, \pi_t)$	R
$p = t + 1$	$y^1 - d_{m+1}^{-1} q^\gamma(\pi_t)$	$(\pi_t, \pi_1, \ldots, \pi_{t-1})$	$R - e(\pi_t)$

TABLE 3

s is the Index of the Vertex of $\tau(w^1, \omega)$ to Be Replaced

	$\bar{w}^1$	$\bar{\omega}$	$\bar{a}$
$s = 1$	$w^1 + d_m^{-1} q^\gamma(\omega_1)$	$(\omega_2, \ldots, \omega_n, \omega_1, \omega_{n+1})$	$a + e(\omega_1)$
$1 < s < n + 1$	w^1	$(\omega_1, \ldots, \omega_{s-2}, \omega_s, \omega_{s-1}, \ldots, \omega_{n+1})$	a
$s = n + 1$	$w^1 - d_m^{-1} q^\gamma(\omega_n)$	$(\omega_n, \omega_1, \ldots, \omega_{n-1}, \omega_{n+1})$	$a - e(\omega_n)$

TABLE 4

s is the Index of the Vertex of τ to Be Replaced

	$c(\bar{\tau})$	$\bar{\delta}$
$s = 1$	$c(\tau) + \delta_1 k_m^{-1}(\bar{w}^{n+1} - w^1)$	$(\delta_2, \ldots, \delta_{n+1}, \delta_1)$
$1 < s < n + 1$	$c(\tau) + \delta_s k_m^{-1}(\bar{w}^s - w^s)$	$(\delta_1, \ldots, \delta_{n+1})$
$s = n + 1$	$c(\tau) + \delta_{n+1} k_m^{-1}(\bar{w}^1 - w^{n+1})$	$(\delta_{n+1}, \delta_1, \ldots, \delta_n)$

$\{\omega_1, \ldots, \omega_{n+1}\}$, are connected with the $n + 1 - t$ vertices w^i, $\omega_i \notin T$, of τ on level m, then the $(n + 1)$-simplices obtained in this way induce a triangulation of $\tau \times [m, m + 1]$.

The proof of this lemma is a straightforward generalization of the theorem on the unit simplex, which proof can be found in van der Laan and Talman [12].

To prove that the union of the triangulation of $\tau(w^1, \omega) \times [m, m + 1]$ over all τ in V_m is a triangulation of $S \times [m, m + 1]$ we need to know how $\tau(w^1, \omega)$ and $c(\tau)$ change when we move from τ to an adjacent simplex $\bar{\tau}$. So, let $\tau(w^1, \omega)$ be an n-simplex in $G_m(\gamma, j_0)$ and let $\bar{\tau}(\bar{w}^1, \bar{\omega})$ be the n-simplex in V_m sharing with τ the facet opposite w^s. If $\bar{\tau}$ lies in $G_m(\gamma, j_0)$, then according to Table 1, $\bar{\tau}$ is obtained from τ as given in Table 3.

The centre point of $\bar{\tau}$ is in this case as given in Table 4.

We will now consider the cases that the facet of τ opposite vertex w^s is either a facet of an n-simplex $\bar{\tau}(\bar{w}^1, \bar{\omega})$ in $G_m(\bar{\gamma}, j_0)$ with $\bar{\gamma} \neq \gamma$ or a facet of an n-simplex $\bar{\tau}(\bar{w}^1, \bar{\omega})$ in $G(\gamma, \bar{j}_0)$ with $\bar{j}_0 \neq j_0$.

The first case occurs iff one of the following two cases occurs, i.e. either

$$1 < s < n + 1, \quad \omega_s = \left(j, k_i^j\right), \quad \omega_{s-1} = \left(j, k_{i-1}^j\right) \quad \text{if } i > 1 \text{ and}$$

$$\omega_{s-1} = Z^0 \quad \text{if } i = 1, \text{ and } a(\omega_{s-1}) = a(\omega_s) \text{ for certain } j \in I_N$$

or

$$s = n + 1, \quad \omega_n = \left(j_0, k_{t(j_0)-1}^{j_0}\right) \text{ and } a(\omega_n) = 0.$$

These subcases coincide with case (b) of Lemma 3.4 for $t = n$ and case (c) of Lemma 3.4 for $t = n$ and $j = j_0$ respectively. In both subcases the parameters $\bar{w}^1$, $\bar{\omega}$ and $\bar{a}$ of

$\bar{\tau}$ are given by

$$\bar{w}^1 = w^1, \quad \bar{a} = a \quad \text{and}$$

$$\bar{\omega} = \begin{cases} (\omega_1, \dots, \omega_{s-2}, \omega_s, \omega_{s-1}, \dots, \omega_{n+1}) & \text{if } i > 1, \\ (\omega_1, \dots, \omega_{s-2}, \bar{Z}^0, (j, k_0^j), \omega_{s+1}, \dots, \omega_{n+1}) & \text{if } i = 1, \end{cases} \tag{4.1}$$

and $\bar{\gamma}$ is given by

$$\bar{\gamma}_h = \begin{cases} ((j, k_0^j), \dots, (j, k_{i-2}^j), (j, k_i^j), (j, k_{i-1}^j), \dots, (j, k_{t(j)}^j)), & h = j, \\ \gamma_h, & h \neq j. \end{cases} \tag{4.2}$$

The second case occurs iff case (c) of Lemma 3.4 holds for $t = n$ and for some $j \neq j_0$, i.e., when

$$s = n + 1, \quad \omega_n = (j, k_{t(j)}^j) \quad \text{for certain } j \in I_N,\ j \neq j_0, \text{ and } a(\omega_n) = 0. \tag{4.3}$$

In this case the parameters $\bar{w}^1$, $\bar{\omega}$ and $\bar{a}$ of $\bar{\tau}$ are given by

$$\bar{w}^1 = w^1, \quad \bar{\omega} = (\omega_1, \dots, \omega_{n-1}, \omega_{n+1}, \omega_n) \quad \text{and } \bar{a} = a \tag{4.4}$$

and $\bar{j}_0$ is given by $\bar{j}_0 = j$. In both cases the centrepoint of $\bar{\tau}$ is given by

$$c(\bar{\tau}) = c(\tau) + \delta_s k_m^{-1}(\bar{w}^s - w^s). \tag{4.5}$$

THEOREM 4.3. *The union of the triangulations of $\tau(w^1, \omega) \times [m, m+1]$ over all n-simplices τ of V_m triangulates $S \times [m, m+1]$.*

PROOF. The triangulation of $\tau \times [m, m+1]$ is well defined for all simplices τ of V_m. Let $\tau(w^1, \omega)$ and $\bar{\tau}(\bar{w}^1, \bar{\omega})$ be two adjacent simplices of V_m and let x be a grid point of V_{m+1} in the common facet. Then it is sufficient to prove that if in the triangulation of $\tau \times [m, m+1]$, x is connected with a vertex w of $\tau \cap \bar{\tau}$, x is also connected with w in the triangulation of $\bar{\tau} \times [m, m+1]$. Suppose that τ lies in $G_m(\gamma, j_0)$ for the permutation γ and index j_0, then we have to consider the following three cases: (a) $\bar{\tau}(\bar{w}^1, \bar{\omega})$ lies in $G_m(\gamma, j_0)$, (b) $\bar{\tau}(\bar{w}^1, \bar{\omega})$ lies in $G_m(\bar{\gamma}, j_0)$ where $\bar{\gamma}$ is given as in (4.2) or (c) $\bar{\tau}(\bar{w}^1, \bar{\omega})$ lies in $G_m(\gamma, j)$, $j \neq j_0$, with j given as in (4.3).

Case (a). $\bar{\tau}(\bar{w}^1, \bar{\omega})$ is an n-simplex in $G_m(\gamma, j_0)$ with a common facet with τ opposite vertex w^s, $1 \leqslant s \leqslant n+1$.

The simplex $\bar{\tau}$ is given as in Table 3 and the centre point of $\bar{\tau}$ is given by (see Table 4)

$$c(\bar{\tau}) = c(\tau) + \delta_s d_{m+1}^{-1}[q^\gamma(\omega_s) - q^\gamma(\omega_{s-1})], \tag{4.6}$$

with the convention $s - 1 = n + 1$ if $s = 1$. Now let T be the subset of $\{\omega_1, \dots, \omega_{n+1}\}$ such that x lies in $\mathring{A}(T, \tau)$, so that

$$x = c(\tau) + \sum_{j \in T} \alpha_j d_{m+1}^{-1} q^\gamma(j), \tag{4.7}$$

for certain positive integers α_j, $j \in T$. Since the point x lies in the facet of τ opposite vertex w^s, (4.7) gives us

$$\delta_s + \alpha_{\omega_{s-1}} - \alpha_{\omega_s} = 0. \tag{4.8}$$

Combining (4.6), (4.7) and (4.8) yields

$$\begin{aligned}
x &= c(\tau) + \sum_{j\in T} \alpha_j d_{m+1}^{-1} q^{\gamma}(j) \\
&= c(\bar{\tau}) - \delta_s d_{m+1}^{-1}[q^{\gamma}(\omega_s) - q^{\gamma}(\omega_{s-1})] + \sum_{j\in T} \alpha_j d_{m+1}^{-1} q^{\gamma}(j) \\
&= c(\bar{\tau}) + \sum_{j\in T\setminus\{\omega_{s-1},\omega_s\}} \alpha_j d_{m+1}^{-1} q^{\gamma}(j) + (\alpha_{\omega_{s-1}} + \delta_s) d_{m+1}^{-1} q^{\gamma}(\omega_{s-1}) \\
&\quad + (\alpha_{\omega_s} - \delta_s) d_{m+1}^{-1} q^{\gamma}(\omega_s) \\
&= c(\bar{\tau}) + \sum_{j\in T} \bar{\alpha}_j d_{m+1}^{-1} q^{\gamma}(j)
\end{aligned}$$

with the coefficients $\bar{\alpha}_j$, $j \in T$ given by

$$\bar{\alpha}_j = \begin{cases} \alpha_j, & j \in T\setminus\{\omega_{s-1}, \omega_s\}, \\ \alpha_{\omega_s}, & j = \omega_{s-1}, \\ \alpha_{\omega_{s-1}}, & j = \omega_s. \end{cases} \tag{4.9}$$

The point x therefore also lies in $\mathring{A}(\bar{T}, \bar{\tau})$ with $\bar{T}$ given by

$$\bar{T} = \begin{cases} T, & \omega_{s-1}, \omega_s \in T \quad \text{or} \quad \omega_{s-1}, \omega_s \notin T, \\ T\setminus\{\omega_{s-s}\} \cup \{\omega_s\}, & \omega_{s-1} \in T, \quad \omega_s \notin T, \\ T\setminus\{\omega_s\} \cup \{\omega_{s-1}\}, & \omega_s \in T, \quad \omega_{s-1} \notin T, \end{cases} \tag{4.10}$$

which proves the theorem for Case (a).

Case (b). $\bar{\tau}(\bar{w}^1, \bar{\omega})$ is an n-simplex in $G_m(\bar{\gamma}, j_0)$, with $\bar{\gamma}$ given as in (4.2), with a common facet with τ opposite vertex w^s, $1 < s \leqslant n+1$.

The simplex $\bar{\tau}$ is given as in (4.1) and the centre point of $\bar{\tau}$ is given by

$$c(\bar{\tau}) = c(\tau) + \delta_s d_{m+1}^{-1}[q^{\gamma}(\omega_s) - q^{\bar{\gamma}}(\bar{\omega}_s)]. \tag{4.11}$$

Combining (4.11), (4.7) and (4.8) yields the following

$$\begin{aligned}
x &= c(\tau) + \sum_{j\in T} \alpha_j d_{m+1}^{-1} q^{\gamma}(j) \\
&= c(\bar{\tau}) - \delta_s d_{m+1}^{-1}[q^{\gamma}(\omega_s) - q^{\bar{\gamma}}(\bar{\omega}_s)] + \sum_{j\in T} \alpha_j d_{m+1}^{-1} q^{\gamma}(j) \\
&= c(\bar{\tau}) + \sum_{j\in T\setminus\{\omega_{s-1},\omega_s\}} \alpha_j d_{m+1}^{-1} q^{\bar{\gamma}}(j) + \delta_s d_{m+1}^{-1} q^{\bar{\gamma}}(\bar{\omega}_s) \\
&\quad + \alpha_{\omega_{s-1}} d_{m+1}^{-1} q^{\gamma}(\omega_{s-1}) + (\alpha_{\omega_s} - \delta_s) d_{m+1}^{-1} q^{\gamma}(\omega_s) \\
&= c(\bar{\tau}) + \sum_{j\in T\setminus\{\omega_{s-1},\omega_s\}} \alpha_j d_{m+1}^{-1} q^{\bar{\gamma}}(j) + \delta_s d_{m+1}^{-1} q^{\bar{\gamma}}(\bar{\omega}_s) \\
&\quad + \alpha_{\omega_{s-1}} d_{m+1}^{-1}[q^{\bar{\gamma}}(\bar{\omega}_{s-1}) + q^{\bar{\gamma}}(\bar{\omega}_s)] \\
&= c(\bar{\tau}) + \sum_{j\in T\setminus\{\omega_{s-1},\omega_s\}} \bar{\alpha}_j d_{m+1}^{-1} q^{\bar{\gamma}}(j) + \bar{\alpha}_{\bar{\omega}_{s-1}} d_{m+1}^{-1} q^{\bar{\gamma}}(\bar{\omega}_{s-1}) \\
&\quad + \bar{\alpha}_{\bar{\omega}_s} d_{m+1}^{-1} q^{\bar{\gamma}}(\bar{\omega}_s)
\end{aligned}$$

where the coefficients $\bar{\alpha}_j$, $j \in T \setminus \{\omega_{s-1}, \omega_s\} \cup \{\bar{\omega}_{s-1}, \bar{\omega}_s\}$ are given by

$$\bar{\alpha}_j = \begin{bmatrix} \alpha_j, & j \in T \setminus \{\omega_{s-1}, \omega_s\}, \\ \alpha_{\omega_{s-1}}, & j = \bar{\omega}_{s-1}, \\ \alpha_{\omega_s}, & j = \bar{\omega}_s. \end{bmatrix}$$

The point x then also lies in $\mathring{A}(\bar{T}, \bar{\tau})$ with $\bar{T}$ given by

$$\bar{T} = \begin{bmatrix} T \setminus \{\omega_{s-1}, \omega_s\} \cup \{\bar{\omega}_{s-1}, \bar{\omega}_s\}, & \omega_{s-1}, \omega_s \in T \quad \text{or} \quad \omega_{s-1}, \omega_s \notin T, \\ T \setminus \{\omega_{s-1}\} \cup \{\bar{\omega}_{s-1}\}, & \omega_{s-1} \in T,\ \omega_s \notin T, \\ T \setminus \{\omega_s\} \cup \{\bar{\omega}_s\}, & \omega_{s-1} \notin T,\ \omega_s \in T \end{bmatrix}$$

which proves the theorem for Case (b).

Case (c). $\bar{\tau}(\bar{w}^1, \bar{\omega})$ is an n-simplex in $G_m(\gamma, j)$, where j is given as in (4.3), having a common facet with τ opposite vertex w^{n+1}.

The simplex $\bar{\tau}$ is given as in (4.4) and the centrepoint of $\bar{\tau}$ is given by

$$c(\bar{\tau}) = c(\tau) + \delta_{n+1} d_{m+1}^{-1}\left[q^{\gamma}(\omega_{n+1}) - q^{\gamma}(\omega_n)\right].$$

This case is similar to Case (a) for $s = n + 1$ and yields the same $\bar{T}$ and $\bar{\alpha}_j$, $j \in T$. ■

The three cases in the proof of Theorem 4.3 are illustrated in Figure 1. Case (a) is illustrated by τ_1 and the vertex $w^3 = w$, Case (b) is illustrated by τ_2 and the vertex $w^2 = w$, and Case (c) is illustrated by τ_3 and the vertex $w^3 = w$ of τ_3.

We have now shown that we can triangulate $S \times [m, m + 1]$ for $m = 1, 2, \ldots$ in $(n + 1)$-simplices with on each level m the V-triangulation with grid size d_m^{-1} as the underlying triangulation. The $(n + 1)$-simplices ψ^{γ} are given by

$$\psi^{\gamma} = \mathrm{co}\Big(\mathrm{co}\big(\{w^i | \omega_i \notin T\}\big) \times \{m\},\ \sigma\big(y^1, \pi(T)\big) \times \{m + 1\}\Big)$$

with $\tau(w^1, \omega)$ an n-simplex of $G_m(\gamma, j_0)$ and $\sigma(y^1, \pi(T))$ a t-simplex in $A(T, \tau)$ on level $m + 1$. Combining the triangulations of $S \times [m, m + 1]$, $m = 1, 2, \ldots$, we obtain a triangulation of $S \times [1, \infty)$. In the following section we will describe how this triangulation induces a triangulation of the boundary faces of $S \times [1, \infty)$ allowing us also to follow the piecewise linear paths of points satisfying (2.4) on the boundary of $S \times [1, \infty)$.

5. Triangulation of the boundary of $S \times [1, \infty)$. In this section we describe in detail how the triangulation of $S \times [1, \infty)$ presented in the previous section triangulates a boundary piece $S(U) \times [m, m + 1]$, $m = 1, 2, \ldots$, where U is a (nonempty) subset of I satisfying $|U_j| \leqslant n_j$ for all $j \in I_N$. The set $S(U)$ is first subdivided in $A(\gamma, j_0)$'s. Let $\gamma_j = ((j, k_0^j), \ldots, (j, k_{t(j)}^j))$ denote a permutation of the elements in $I(j) \setminus U_j$ where $t(j) = n_j - u_j$, $u_j = |U_j|$, and let $Z_j^0 = \{(j, k_0^j)\}$, $Z_j = \{(j, k_1^j), \ldots, (j, k_{t(j)}^j)\}$, $j \in I_N$, $\gamma = (\gamma_1, \ldots, \gamma_N)$, $Z^0 = \bigcup_j Z_j^0$ and $Z = \bigcup_j Z_j$.

DEFINITION 5.1. Let the set U and the permutation vector γ be as defined above. For $j_0 \in I_N$, the set $A(\gamma, j_0)$ is given by

$$A(\gamma, j_0) = \Big\{x \in S | x = v(U) + \alpha(Z^0) q^{\gamma}(Z^0) + \sum_{(j,k) \in Z} \alpha(j, k) q^{\gamma}(j, k),$$

$$\text{where } 0 \leqslant \alpha\big(j, k_{t(j)}^j\big) \leqslant \cdots \leqslant \alpha\big(j, k_1^j\big) \leqslant \alpha(Z^0) \leqslant 1,\ j \in I_N$$

$$\text{and } \alpha\big(j_0, k_{t(j_0)}^{j_0}\big) = 0\Big\}$$

where $v(U)$ is the relative projection of v on $S(U)$, i.e.

$$v_{i,h}(U) = \begin{bmatrix} v_{i,h}\left(\sum_{(i,k)\notin U_i} v_{i,k}\right)^{-1}, & (i,h)\notin U_i, \\ 0, & (i,h)\in U_i, \end{bmatrix} \qquad i\in I_N,$$

where

$$q_j^{\gamma}(Z^0) = p_j\left(Z_j^0\right) - p_j\left(Z_j^0 \cup Z_j\right), \qquad j\in I_N,$$

and where the $(N+n)$-vector $q^{\gamma}(j, k_i^j)$, $i = 1,\dots, t(j)$, $j\in I_N$, is given by $q_h^{\gamma}(j, k_i^j) = 0$, $h \neq j$, and

$$q_j^{\gamma}(j, k_i^j) = p_j(\{(j,k_0^j),\dots,(j,k_i^j)\}) - p_j(\{(j,k_0^j),\dots,(j,k_{i-1}^j)\}).$$

Observe that Definition 5.1 coincides with Definition 3.1 if U is empty. The set $A(\gamma, j_0)$ is a $\Sigma_{j=1}^N(n_j - u_j)$-dimensional subset of $S(U)$. Let $A(\gamma)$ be the union of $A(\gamma, j_0)$ over all indices $j_0 \in I_N$ and let $u = \Sigma_{j=1}^N u_j$. Recall that $n = \Sigma_{j=1}^N n_j$. Then each set $A(\gamma, j_0)$ is triangulated by the V-triangulation of S with grid size d_m^{-1} by $G_m(\gamma, j_0)$.

DEFINITION 5.2. The set $G_m(\gamma, j_0)$ is the collection of $(n-u)$-simplices $\tau(w^1, \omega)$ with vertices $w^1,\dots, w^{n-u+1}$ such that

(1) $w^1 = v(U) + a(Z^0)d_m^{-1}q^{\gamma}(Z^0) + \Sigma_{(j,k)\in Z}a(j,k)d_m^{-1}q^{\gamma}(j,k)$, for nonnegative integers $a(Z^0)$ and $a(j,k)$, $(j,k)\in Z$, such that for all $j\in I_N$, $0 \leqslant a(j, k_{t(j)}^j) \leqslant \cdots \leqslant a(j,k_1^j) \leqslant a(Z^0) \leqslant d_m - 1$ and $a(j_0, k_{t(j_0)}^{j_0}) = 0$;

(2) $\omega = (\omega_1,\dots,\omega_{n-u+1})$ is a permutation of the elements consisting of Z^0 and the $n-u$ elements of Z such that $\omega_{n-u+1} = (j_0, k_{t(j_0)}^{j_0})$, and for all $i = 1,\dots, t(j)$: $s > s'$ if $a(j,k_i^j) = a(j,k_{i-1}^j)$, where $\omega_s = (j,k_i^j)$ and $\omega_{s'} = (j,k_{i-1}^j)$ if $i > 1$ and $\omega_{s'} = Z^0$ if $i = 1$, $j\in I_N$;

(3) $w^{i+1} = w^i + d_m^{-1}q^{\gamma}(\omega_i)$, $i = 1,\dots, n-u+1$, with the convention $i+1 = 1$ in the case $i = n-u+1$.

It is clear that $G_m(\gamma, j_0)$ is a triangulation of $A(\gamma, j_0)$ and that the union $G_m(\gamma)$ of $G_m(\gamma, j_0)$ over all $j_0\in I_N$ triangulates $A(\gamma)$. Finally we observe that the union $G_m(U)$ of $G_m(\gamma)$ over all permutation vectors γ of the elements in $I \setminus U$ induces a triangulation of $S(U)$. Some $G_m(\gamma, j_0)$ when U is $\{(1,2)\}$ and $\{(2,1)\}$ are illustrated in Figure 4 if $N = 2$, $n_1 = 1$, $n_2 = 2$ and $d_m = 1$. The arrows on the edges determine the ordering of the vertices in the simplices τ of $G_m(\gamma, j_0)$.

Notice that the boundary face $S(U)$ of S is in fact equivalent to the set $\prod_{j=1}^N S^{n_j - u_j}$. The replacement rules on $S(U)$ are therefore similar to the ones described in the previous section on S.

As in §4, given $\theta_1^m,\dots,\theta_{n+1}^m$, we define for each $(n-u)$-simplex $\tau(w^1,\omega)$ in $G_m(U)$ the centre point $c(\tau)$ of τ as $c(\tau) = \Sigma_{i=1}^{n-u+1}\delta_i k_m^{-1}w^i$ where the vector $\delta = (\delta_1,\dots,\delta_{n-u+1})$ is given by

$$\delta_i = \begin{bmatrix} \theta_{s^m(w^i)}^m, & i\neq r+1, \\ k_m - \sum_{\substack{i=1\\ i\neq r+1}}^{n-u+1}\theta_{s^m(w^i)}^m, & i = r+1, \end{bmatrix} \quad \text{with } \omega_r = Z^0.$$

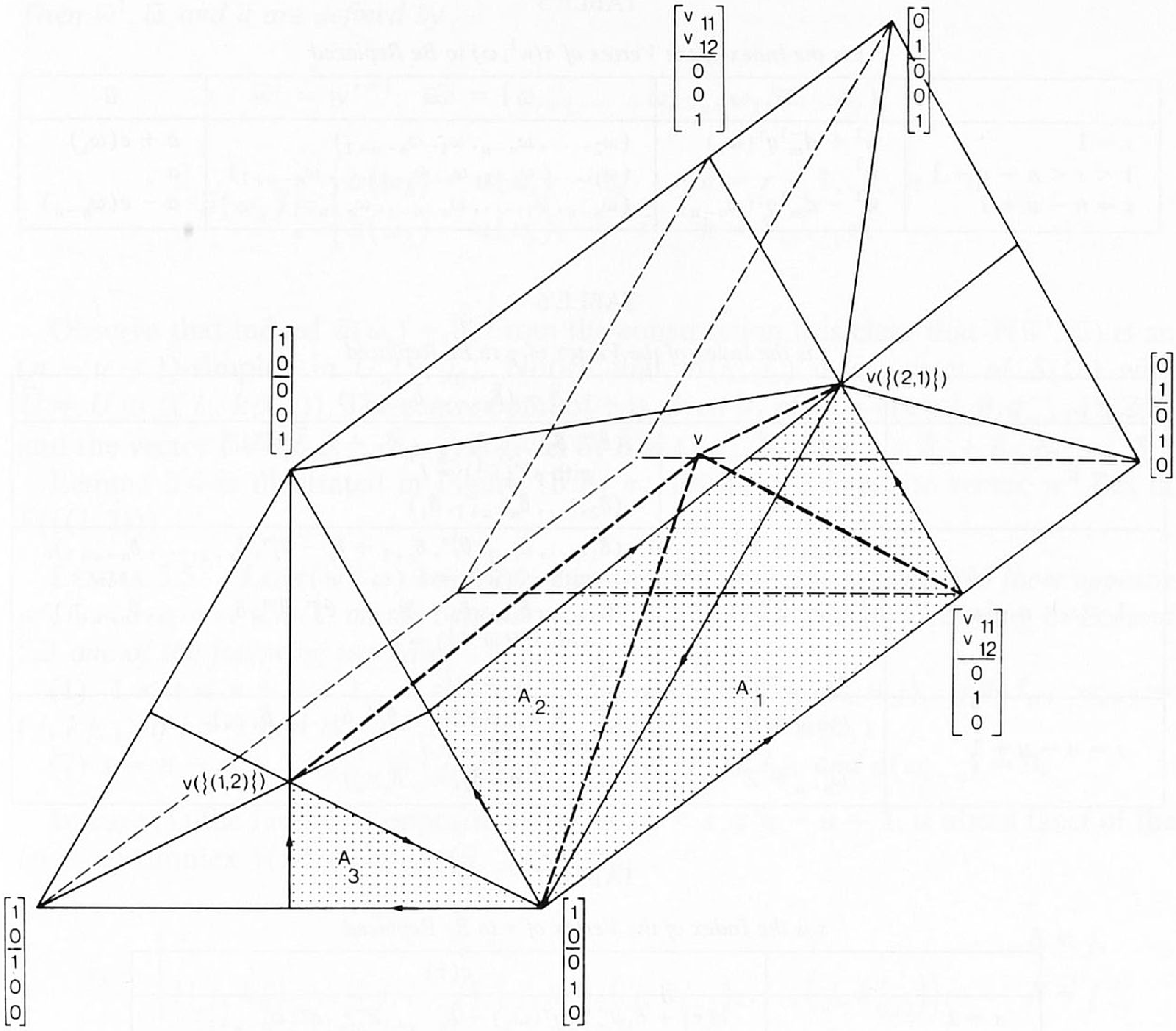


FIGURE 4. Some regions $A(\gamma, j_0)$; $A_1 = A(\gamma^1, 2)$ in $S(\{(2,1)\})$ with $\gamma^1 = (((1,1),(1,2)), ((2,2),(2,3)))$, $A_2 = A(\gamma^1, 1)$ in $S(\{(2,1)\})$, and $A_3 = A(\gamma^2, 2)$ in $S(\{(1,2)\})$ with $\gamma^2 = (((1,1)),((2,2),(2,1),(2,3)))$, $N = 2$, $n_1 = 1$, $n_2 = 2$.

Observe that $\sum_{i=1}^{n-u+1}\delta_i$ is equal to k_m and that $c(\tau)$ is a grid point of V_{m+1} in $\tau(w^1, \omega)$. Furthermore for $U = \varnothing$ the centre point coincides with the centre point defined in §4.

Since the algorithm will move from one simplex to an adjacent one we have to describe how the representation of the latter one can be obtained from the representation of the former one, and how the centre point changes from one simplex to another adjacent simplex.

So let $\tau(w^1, \omega)$ and $\bar{\tau}(\bar{w}^1, \bar{\omega})$ be in some $G_m(\gamma, j_0)$ with a common facet opposite vertex w^s, $1 \leqslant s \leqslant n - u + 1$, then $\bar{\tau}$ can be obtained from τ as given in Table 5. Furthermore, in Tables 6 and 7 we describe how $\bar{\delta}$ and $c(\bar{\tau})$ are obtained from δ and $c(\tau)$. In the next lemma the conditions are given when a facet of an $(n-u)$-simplex $\tau(w^1, \omega)$ in $G_m(\gamma, j_0)$ lies in the boundary of $A(\gamma, j_0)$, see also Lemma 3.4.

LEMMA 5.3. *Let $\tau(w^1, \omega)$ be an $(n-u)$-simplex in $G_m(\gamma, j_0)$, then the facet opposite vertex w^s, $1 \leqslant s \leqslant n-u+1$, lies on the boundary of $A(\gamma, j_0)$ iff*

(a) $s = 1$, $\omega_1 = Z^0$ *and* $a(Z^0) = d_m - 1$.

(b) $1 < s \leqslant n - u + 1$, $\omega_s = (j, k_i^j)$ *for certain* $1 \leqslant i \leqslant t(j)$, $j \in I_N$, $\omega_{s-1} = (j, k_{i-1}^j)$ *if* $i > 1$ *and* $\omega_{s-1} = Z^0$ *if* $i = 1$, *and* $a(\omega_{s-1}) = a(\omega_s)$.

(c) $s = n - u + 1$, $\omega_{n-u} = (j, k_{t(j)}^j)$ *for certain* $j \in I_N$ *and* $a(\omega_{n-u}) = 0$.

The lemma follows immediately from the definitions of $G_m(\gamma, j_0)$ and $A(\gamma, j_0)$. If the facet of $\tau(w^1, \omega)$ opposite vertex w^s, $1 \leqslant s \leqslant n-u+1$, lies on the boundary of

TABLE 5

s is the Index of the Vertex of $\tau(w^1, \omega)$ to Be Replaced

	$\bar{w}^1$	$\bar{\omega}$	$\bar{a}$
$s = 1$	$w^1 + d_m^{-1} q^\gamma(\omega_1)$	$(\omega_2, \ldots, \omega_{n-u}, \omega_1, \omega_{n-u+1})$	$a + e(\omega_1)$
$1 < s < n - u + 1$	w^1	$(\omega_1, \ldots, \omega_{s-2}, \omega_s, \omega_{s-1}, \ldots, \omega_{n-u+1})$	a
$s = n - u + 1$	$w^1 - d_m^{-1} q^\gamma(\omega_{n-u})$	$(\omega_{n-u}, \omega_1, \ldots, \omega_{n-u-1}, \omega_{n-u+1})$	$a - e(\omega_{n-u})$

TABLE 6

s is the Index of the Vertex of τ to Be Replaced

		$\bar{\delta} = (\bar{\delta}_1, \ldots, \bar{\delta}_{n-u+1})$
$s = 1$	$\omega_1 = Z^0$	$(\theta_l^m, \delta_3, \ldots, \delta_{n-u+1}, \delta_1 + \delta_2 - \theta_l^m)$, with $s^m(\bar{w}^1) = l$
	$\omega_1 \neq Z^0$	$(\delta_2, \ldots, \delta_{n-u+1}, \delta_1)$
$1 < s < n - u + 1$	$\omega_{s-1} = Z^0$	$(\delta_1, \ldots, \delta_{s-1}, \theta_l^m, \delta_{s+1} + \delta_s - \theta_l^m, \delta_{s+2}, \ldots, \delta_{n-u+1})$, with $s^m(\bar{w}^s) = l$
	$\omega_s = Z^0$	$(\delta_1, \ldots, \delta_{s-1}, \delta_s + \delta_{s+1} - \theta_l^m, \theta_l^m, \delta_{s+2}, \ldots, \delta_{n-u+1})$, with $s^m(\bar{w}^{s+1}) = l$
	$\omega_{s-1}, \omega_s \neq Z^0$	$(\delta_1, \ldots, \delta_{n-u+1})$
$s = n - u + 1$	$\omega_{n-u} = Z^0$	$(\theta_l^m, \delta_{n-u+1} + \delta_1 - \theta_l^m, \delta_2, \ldots, \delta_{n-u})$, with $s^m(\bar{w}^1) = l$
	$\omega_{n-u} \neq Z^0$	$(\delta_{n-u+1}, \delta_1, \delta_2, \ldots, \delta_{n-u})$

TABLE 7

s is the Index of the Vertex of τ to Be Replaced

	$c(\bar{\tau})$
$s = 1$	$c(\tau) + \delta_1 d_{m+1}^{-1} q^\gamma(\omega_1) - \bar{\delta}_{n-u+1} d_{m+1}^{-1} q^\gamma(\omega_{n-u+1})$
$1 < s < n - u + 1$	$c(\tau) + \delta_s d_{m+1}^{-1} q^\gamma(\omega_s) - \bar{\delta}_s d_{m+1}^{-1} q^\gamma(\omega_{s-1})$
$s = n - u + 1$	$c(\tau) + \delta_{n-u+1} d_{m+1}^{-1} q^\gamma(\omega_{n-u+1}) - \bar{\delta}_1 d_{m+1}^{-1} q^\gamma(\omega_{n-u})$

$A(\gamma, j_0)$ then either this facet is an $(n - u - 1)$-simplex in $G_m(U \cup \{(i, h)\})$ for some $(i, h) \notin U$ or it is a facet of another $(n - u)$-simplex $\bar{\tau}(\bar{w}^1, \bar{\omega})$ in $G_m(U)$ with either $\bar{\tau}$ in $G_m(\bar{\gamma}, j_0)$, $\bar{\gamma} \neq \gamma$, or $\bar{\tau}$ in $G(\gamma, \bar{j}_0)$, $\bar{j}_0 \neq j_0$, as shown in the next two lemmas.

LEMMA 5.4. *Let $\tau(w^1, \omega)$ be an $(n - u)$-simplex in $G_m(\gamma, j_0)$ with $\bar{\tau}$ the facet opposite vertex w^1 on the boundary of $A(\gamma, j_0)$, then $\bar{\tau}$ is the $(n - u - 1)$-simplex $\bar{\tau}(\bar{w}^1, \bar{\omega})$ in the subset $A(\bar{\gamma}, \bar{j}_0)$ of $S(U \cup \{(j_0, k_{t(j_0)}^{j_0})\})$ where*

$$\bar{\gamma}_j = \begin{cases} \gamma_j, & j \neq j_0, \\ \left(\left(j_0, k_0^{j_0}\right), \ldots, \left(j_0, k_{t(j_0)-1}^{j_0}\right)\right), & j = j_0 \quad \textit{and} \end{cases}$$

$$\bar{Z}_{j_0} = Z_{j_0} \Big\backslash \left\{\left(j_0, k_{t(j_0)}^{j_0}\right)\right\} \quad \textit{and} \quad \bar{Z}_j = Z_j, \qquad j \neq j_0.$$

Furthermore let J be the index set such that $j \in J$ if both $j \neq j_0$ and $t(j) \geq 1$, and $j_0 \in J$ if $t(j_0) \geq 2$. Let $r(j)$, $j \in J$, be such that $\omega_{r(j)} = (j, k_{t(j)}^j)$ if $j \neq j_0$ and $\omega_{r(j_0)} = (j_0, k_{t(j_0)-1}^{j_0})$ if $j_0 \in J$. The index r is now given by

$$r = r(\bar{j}_0) = \max\left\{r(j) | a(\omega_{r(j)}) = \min\left\{a(\omega_{r(i)}) | i \in J\right\}\right\}.$$

Then $\bar{w}^1$, $\bar{\omega}$ *and* $\bar{a}$ *are defined by*

$$\bar{w}^1 = w^{r+1}, \quad \bar{\omega} = (\omega_{r+1}, \dots, \omega_{n-u}, \omega_1, \dots, \omega_r),$$

$$\bar{a}(\omega_h) = \begin{bmatrix} a(\omega_h) - a(\omega_r) - 1, & h = r+1, \dots, n-u, \\ a(\omega_h) - a(\omega_r), & h = 1, \dots, r. \end{bmatrix}$$

Observe that indeed $\bar{a}(\omega_r) = 0$. From the construction it is clear that $\bar{\tau}(\bar{w}^1, \bar{\omega})$ is an $(n-u-1)$-simplex in $G_m(\bar{\gamma}, \bar{j}_0)$. Notice that $A(\bar{\gamma}, \bar{j}_0)$ is a subset of $S(\bar{U})$ with $\bar{U} = U \cup \{(j_0, k^{j_0}_{t(j_0)})\}$. The centre point of τ is given by $c(\bar{\tau}) = c(\tau) + \delta_1 d^{-1}_{m+1} q^{\gamma}(Z^0)$, and the vector $\bar{\delta} = (\bar{\delta}_1, \dots, \bar{\delta}_{n-u})$ is given by $\bar{\delta} = (\delta_{r+1}, \dots, \delta_{n-u+1}, \delta_1 + \delta_2, \delta_3, \dots, \delta_r)$.

Lemma 5.4 is illustrated in Figure 1b by τ_3. Its facet $\bar{\tau}$ opposite vertex w^1 lies in $S(\{(1,2)\})$.

LEMMA 5.5. *Let* $\tau(w^1, \omega)$ *be an* $(n-u)$*-simplex in* $G_m(\gamma, j_0)$ *with the facet opposite* w^s, $1 < s \leqslant n-u+1$, *on the boundary of* $A(\gamma, j_0)$. *Then we have according to Lemma* 5.3 *one of the following two cases*

(1) $1 < s \leqslant n-u+1$, $\omega_s = (j, k^j_i)$ *for certain* $1 \leqslant i \leqslant t(j)$, $j \in I_N$, $\omega_{s-1} = (j, k^j_{i-1})$ *if* $i > 1$ *and* $\omega_{s-1} = Z^0$ *if* $i = 1$, *and* $a(\omega_{s-1}) = a(\omega_s)$

(2) $s = n-u+1$, $\omega_{n-u} = (j, k^j_{t(j)})$, *for certain* $j \in I_N$, *and* $a(\omega_{n-u}) = 0$.

In case (1) the facet of τ opposite vertex w^s, $1 < s \leqslant n-u+1$, is also a facet of the $(n-u)$-simplex $\bar{\tau}(\bar{w}^1, \bar{\omega})$ in $G_m(\bar{\gamma}, j_0)$ where

$$\bar{\gamma}_h = \begin{bmatrix} \gamma_h, & h \neq j, \\ \left((j, k^j_0), \dots, (j, k^j_{i-1}), (j, k^j_i), (j, k^j_{i-1}), \dots, (j, k^j_{t(j)})\right), & h = j \end{bmatrix}$$

$$\bar{w}^1 = w^1, \quad \bar{a} = a, \text{ and}$$

$$\bar{\omega} = \begin{bmatrix} \left(\omega_1, \dots, \omega_{s-2}, \bar{Z}^0, (j, k^j_0), \omega_{s+1}, \dots, \omega_{n-u+1}\right) & \text{if } i = 1, \\ \left(\omega_1, \dots, \omega_{s-2}, \omega_s, \omega_{s-1}, \omega_{s+1}, \dots, \omega_{n-u+1}\right) & \text{if } i > 1. \end{bmatrix}$$

The centre point of $\bar{\tau}$ is given by

$$c(\bar{\tau}) = c(\tau) + \delta_s d^{-1}_{m+1}\left(q^{\gamma}(\omega_s) - q^{\bar{\gamma}}(\bar{\omega}_s)\right)$$

and $\bar{\delta} = \delta$.

In case (2) the facet of τ opposite vertex w^{n-u+1} is a facet of the $(n-u)$-simplex $\bar{\tau}(\bar{w}^1, \bar{\omega})$ in $G_m(\gamma, \bar{j}_0)$ with $\bar{j}_0 = j$, and with $\bar{w}^1 = w^1$, $\bar{\omega} = (\omega_1, \dots, \omega_{n-u-1}, \omega_{n-u+1}, \omega_{n-u})$ and $\bar{a} = a$.

The centre point of $\bar{\tau}$ is given by

$$c(\bar{\tau}) = c(\tau) + \delta_{n-u+1} d^{-1}_{m+1}\left[q^{\gamma}(\omega_{n-u+1}) - q^{\gamma}(\omega_{n-u})\right],$$

and $\bar{\delta} = \delta$.

Finally, let $\tau(w^1, \omega)$ be an $(n-u)$-simplex in $S(U)$ for some U with $u \geqslant 1$, then τ is a facet of exactly one $(n-u+1)$-simplex in $S(U \setminus \{(i,h)\})$ for any $(i,h) \in U$.

LEMMA 5.6. *Let* $\tau(w^1, \omega)$ *be an* $(n-u)$*-simplex in* $G_m(\gamma, j_0)$ *with* $u \geqslant 1$ *and let* (i, h) *be an element in* U. *Then* τ *is a facet of exactly one* $(n-u+1)$*-simplex* $\bar{\tau}$ *in*

$G_m(U\setminus\{(i,h)\})$. *More precisely*, $\bar\tau$ *lies in* $G_m(\bar\gamma, i)$ *with*

$$\bar\gamma_j = \begin{cases} \big((i,k_0^i),\dots,(i,k_{t(i)}^i),(i,h)\big), & j = i,\\ \gamma_j, & j \neq i.\end{cases}$$

The parameters of the simplex $\bar\tau$ *are given by*

$$\bar w^1 = w^{r+1} - d_m^{-1} q^{\bar\gamma}(Z^0), \qquad \bar\omega = \big(\omega_r,\dots,\omega_{n-u+1},\omega_1,\dots,\omega_{r-1},(i,h)\big),$$

where r *is the index such that* $\omega_r = Z^0$, $\bar a(Z^0) = d_m - 1$, $\bar a(i,h) = 0$, *and the coefficients* $\bar a(j,k)$, $(j,k)\in Z$, *are given by*

$$\bar a(\omega_k) = \begin{cases} d_m - a(\omega_r) + a(\omega_k) - 1, & k = r+1,\dots,n-u+1,\\ d_m - a(\omega_r) + a(\omega_k), & k = 1,\dots,r-1.\end{cases}$$

The centre point of $\bar\tau$ *is given by* $c(\bar\tau) = c(\tau) - \bar\delta_1 d_{m+1}^{-1} q^{\bar\gamma}(Z^0)$ *where the vector* $\bar\delta = (\bar\delta_1,\dots,\bar\delta_{n-u+2})$ *is given by*

$$\bar\delta = (\theta_l^m, \delta_{r+1} - \theta_l^m, \delta_{r+2},\dots,\delta_{n-u+1},\delta_1,\dots,\delta_r), \quad \textit{with } l = s^m(\bar w^1).$$

Lemma 5.6 is illustrated in Figure 1 for $\bar\tau$ in $S(\{(1,2)\})$. The triangulation of $S(U)\times[m, m+1]$ for proper subsets U of I is as follows. Let $\tau(w^1,\omega)$ be an $(n-u)$-simplex in $G_m(\gamma, j_0)$ and let $A(T,\tau)$ for proper subsets T of $\{\omega_1,\dots,\omega_{n-u+1}\}$ be defined as in §4, then on level $m+1$ $A(T,\tau)$ is triangulated by the V-triangulation of S on level $m+1$ in t-simplices $\sigma(y^1,\pi(T))$ with vertices $y^1,\dots,y^{t+1}$ such that

(1) $y^1 = c(\tau) + \sum_{h=1}^{n-u+1} R_{\omega_h} d_{m+1}^{-1} q^{\gamma}(\omega_h) = \sum_{h=1}^{n-u+1}(\delta_h - R_{\omega_h} + R_{\omega_{h-1}}) k_m^{-1} w^h$, $R_{\omega_h} \geq 0$, $\omega_h \in T$, and $R_{\omega_h} = 0$, $\omega_h \notin T$;

(2) $\pi(T) = (\pi_1,\dots,\pi_t)$ is a permutation of the t elements in T;

(3) $y^{i+1} = y^i + d_{m+1}^{-1} q^{\gamma}(\pi_i) = y^i + k_m^{-1}(w^{s+1} - w^s)$ with s such that $\omega_s = \pi_i$, $i = 1,\dots,t$.

The boundary of $A(T,\tau)$ is now given by

$$\operatorname{bd} A(T,\tau) = \bigcup_{\omega_s\in T} (A(T,\tau)\cap\tau_s) \cup \Big(\bigcup_{\omega_s\in T} A(T\setminus\{\omega_s\},\tau)\Big).$$

Note the similarity with the sets $A(T,\tau)$ if U is empty.

The triangulation of $\tau\times[m, m+1]$ is induced by connecting all the simplices $\sigma(y^1,\pi(T))$ in $A(T,\tau)$ on level $m+1$ with the vertices w^i, $\omega_i \notin T$, of τ on level m. An $(n-u+1)$-simplex ψ^γ of this triangulation is thus given by

$$\psi^\gamma = \operatorname{co}\big(\operatorname{co}(\{w^i | \omega_i \notin T\}) \times \{m\}, \sigma(y^1,\pi(T)) \times \{m+1\}\big). \tag{5.1}$$

Similar to Lemma 5.6 we will now show that an $(n-u+1)$-simplex ψ^γ of the triangulation of $S(U)\times[m, m+1]$ for nonempty U is a facet of just one $(n-u+2)$-simplex of the triangulation of $S(U\setminus\{(i,h)\})\times[m, m+1]$, for any $(i,h)\in U$.

LEMMA 5.7. *Let* ψ^γ *be an* $(n-u+1)$*-simplex of the triangulation of* $S(U)\times[m, m+1]$ *with* U *nonempty. Let* (i,h) *be an element of* U, *then* ψ^γ *is a facet of exactly one* $(n-u+2)$*-simplex of the triangulation of* $S(U\setminus\{(i,h)\})\times[m, m+1]$. *Let* $\bar\psi^{\bar\gamma}$ *be this* *simplex, then* $\bar{\gamma}$ *is given by*

$$\bar{\gamma}_j = \begin{cases} \left(\left(i, k_0^i\right), \ldots, \left(i, k_{t(i)}^i\right), (i, h)\right), & j = i, \\ \gamma_j, & j \neq i. \end{cases}$$

The simplex $\bar{\psi}^{\bar{\gamma}}$ is given by

$$\bar{\psi}^{\bar{\gamma}} = \mathrm{co}\left(\mathrm{co}\left(\{\bar{w}^i | \bar{\omega}_i \notin \bar{T}\}\right) \times \{m\}, \bar{\sigma}(\bar{y}^1, \bar{\pi}(\bar{T})) \times \{m+1\}\right)$$

where $\bar{\tau}(\bar{w}^1, \bar{\omega})$ is the $(n-u+1)$-simplex of $G_m(\bar{\gamma}, i)$ defined in Lemma 5.6 and where $\bar{\sigma}$ is determined as follows. Let r be the index such that $\omega_r = Z^0$ and recall that the centre point of $\bar{\tau}$ is given by $c(\bar{\tau}) = c(\tau) - \bar{\delta}_1 d_{m+1}^{-1} q^{\bar{\gamma}}(Z^0)$.

We consider the two cases (i) $\omega_r \notin T$ and (ii) $\omega_r \in T$.
In case (i) the vertex y^1 of $\sigma(y^1, \pi(T))$ is equal to

$$\begin{aligned} y^1 &= c(\tau) + \sum_{h \in T} R_h d_{m+1}^{-1} q^{\gamma}(h) \\ &= c(\bar{\tau}) + \bar{\delta}_1 d_{m+1}^{-1} q^{\bar{\gamma}}(Z^0) + \sum_{h \in T} R_h d_{m+1}^{-1} q^{\bar{\gamma}}(h). \end{aligned}$$

Hence, if $\bar{\delta}_1$ is positive, $\bar{\sigma}$ is the $(t+1)$-simplex $\bar{\sigma}(\bar{y}^1, \bar{\pi}(\bar{T}))$ in $A(\bar{T}, \bar{\tau})$ where

$$\bar{y}^1 = y^1 - d_{m+1}^{-1} q^{\bar{\gamma}}(Z^0), \quad \bar{T} = T \cup \{\omega_r\},$$

$$\bar{\pi}(\bar{T}) = \left(Z^0, \pi_1, \ldots, \pi_t\right), \quad \text{and} \quad \bar{R} = R + \left(\bar{\delta}_1 - 1\right) e(Z^0).$$

If $\bar{\delta}_1 = 0$, then $\bar{\sigma}$ is equal to σ, being a t-simplex in $A(\bar{T}, \bar{\tau})$ with $\bar{T} = T$. In case (ii) we have

$$\begin{aligned} y^1 &= c(\tau) + \sum_{h \in T} R_h d_{m+1}^{-1} q^{\gamma}(h) \\ &= c(\bar{\tau}) + \bar{\delta}_1 d_{m+1}^{-1} q^{\bar{\gamma}}(Z^0) + \sum_{h \in T \setminus \{\omega_r\}} R_h d_{m+1}^{-1} q^{\bar{\gamma}}(h) \\ &\quad + R_{\omega_r} d_{m+1}^{-1} \left[q^{\bar{\gamma}}(Z^0) + q^{\bar{\gamma}}(i, h)\right]. \end{aligned}$$

Hence, the $(t+1)$-simplex $\bar{\sigma}(\bar{y}^1, \bar{\pi}(\bar{T}))$ is given by

$$\bar{y}^1 = y^1, \quad \bar{T} = T \cup \{(i, h)\}, \quad \bar{\pi}(\bar{T}) = \left(\pi_1, \ldots, \pi_{s-1}, (i, h), \pi_s, \ldots, \pi_t\right),$$

where s is the index such that $\pi_s = \omega_r$, and $\bar{R} = R + \bar{\delta}_1 e(Z^0) + R_{\omega_r} e(i, h)$.

From the construction it is clear that $\bar{\sigma}$ lies in $A(\bar{T}, \bar{\tau})$. In all cases the $(n-u+1)$-simplex ψ^{γ} is a facet of the $(n-u+2)$-simplex $\bar{\psi}^{\bar{\gamma}}$. More precisely, if $\omega_r \notin T$ and $\bar{\delta}_1 > 0$, then ψ^{γ} is the facet of $\bar{\psi}^{\bar{\gamma}}$ opposite vertex $(\bar{y}^1, m+1)$. If $\omega_r \notin T$ and $\bar{\delta}_1 = 0$, then ψ^{γ} is the facet of $\bar{\psi}^{\bar{\gamma}}$ opposite vertex $(\bar{w}^1, m)$, and if $\omega_r \in T$, then ψ^{γ} is the facet of $\bar{\psi}^{\bar{\gamma}}$ opposite vertex $(\bar{y}^{s+1}, m+1)$ where $\pi_s = \omega_r$.

By generalizing this lemma it can easily be shown that for any complete extension $\bar{\gamma}$ of γ each $(n-u+1)$-simplex ψ^{γ} of the triangulation of $S(U) \times [m, m+1]$ is a face of just one $(n+1)$-simplex $\bar{\psi}^{\bar{\gamma}}$ of the triangulation of $S \times [m, m+1]$ described in §4.

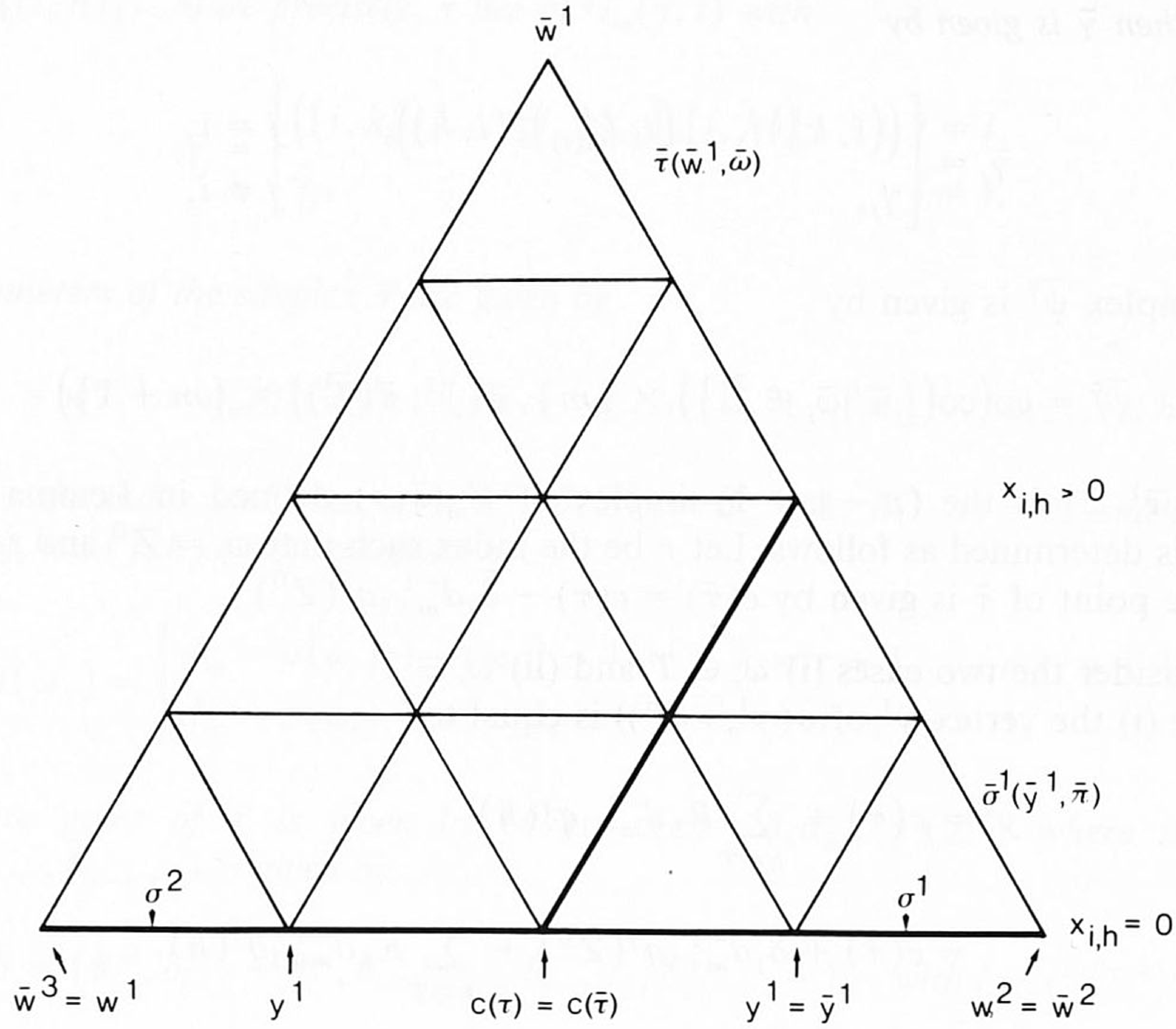


FIGURE 5. Illustration of Lemma 5.7 in the Case $c(\tau) = c(\bar{\tau})$.

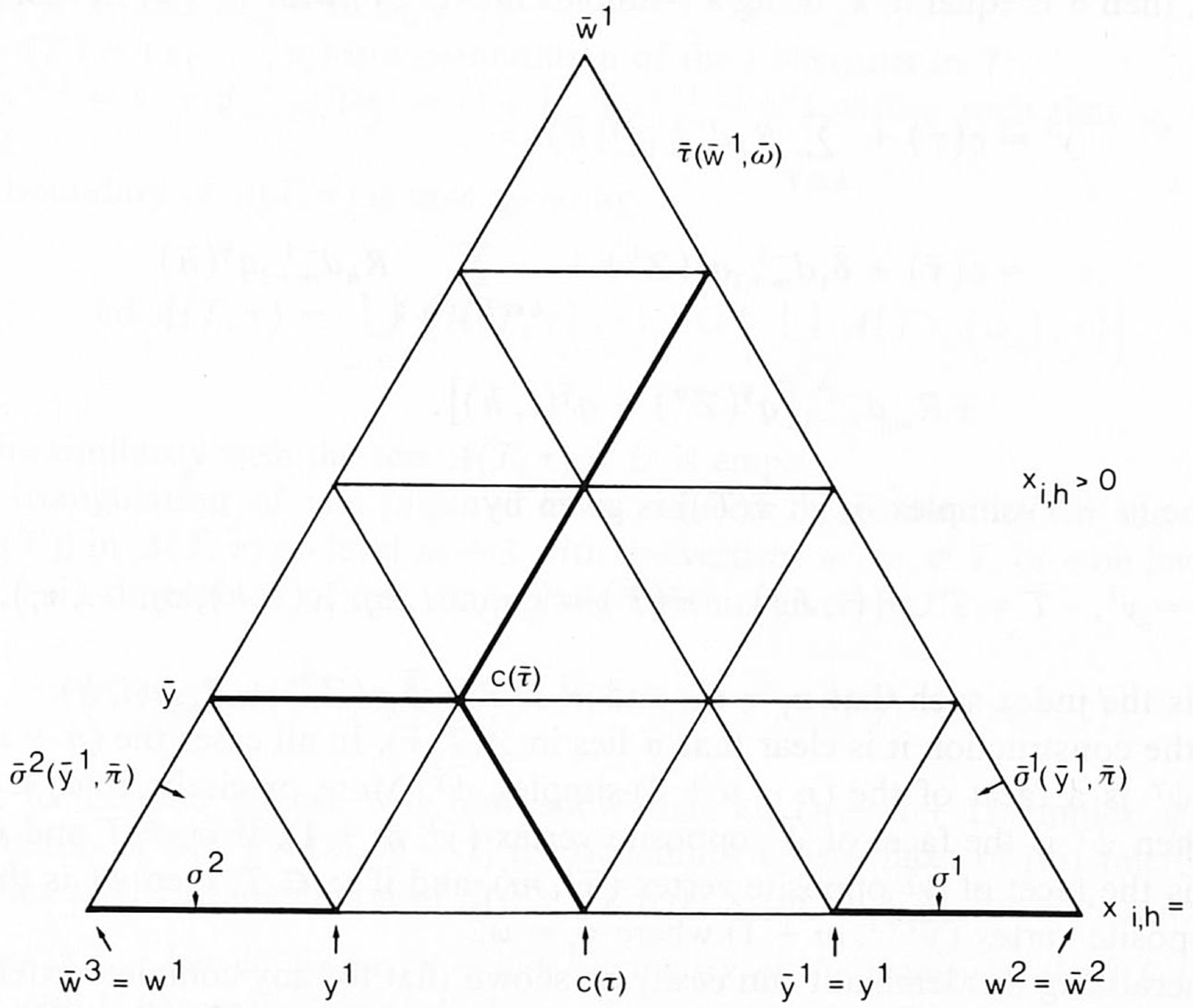


FIGURE 6. Illustration of Lemma 5.7 in the Case $c(\tau) \neq c(\bar{\tau})$.

Lemma 5.7 is illustrated in the Figures 5 and 6. In Figure 5 τ is equal to the 1-dimensional simplex $\tau(w^1, w^2) = \tau(w^1, (Z^0, \omega_2))$ in $S(\{(i, h)\})$. The 2-dimensional simplex $\bar{\tau}$ in S is given by $\bar{\tau}(\bar{w}^1, \bar{w}^2, \bar{w}^3) = \bar{\tau}(\bar{w}^1, (Z^0, \bar{\omega}_2, \bar{\omega}_3))$, $\bar{\omega}_3 = (i, h)$, and $c(\tau) = c(\bar{\tau})$. The 1-simplex $\sigma^1(y^1, (Z^0))$ lies in $A(\{Z^0\}, \tau)$ so that ψ^γ is given by $\mathrm{co}(\{w^2\} \times \{m\}, \sigma^1 \times \{m+1\})$. The latter simplex is a facet of the 3-simplex $\mathrm{co}(\{\bar{w}^2\} \times \{m\}, \bar{\sigma}^1 \times \{m+1\})$ where $\bar{\sigma}^1$ is given by $\bar{\sigma}^1(\bar{y}^1, \bar{\pi})$ with $\bar{y}^1 = y^1$ and $\bar{\pi} = ((i, h), Z^0)$. This coincides with case (ii) of Lemma 5.7. The same holds for $\bar{\sigma}^1(\bar{y}^1, \bar{\pi})$ in Figure 6. The 1-simplex $\sigma^2(y^1, (\omega_2))$ lies in $A(\{\omega_2\}, \tau)$ so that ψ^γ is given by $\mathrm{co}(\{w^1\} \times \{m\}, \sigma^2 \times \{m+1\})$. In Figure 5 we have for σ^2 that Z^0 does not lie in T, so that case (i) of Lemma 5.7 holds. In Figure 5, where $\bar{\delta}_1 = 0$, ψ^γ is a facet of the 3-simplex $\mathrm{co}(\mathrm{co}(\{\bar{w}^1, \bar{w}^3\}) \times \{m\}, \sigma^2 \times \{m+1\})$ and σ^2 lies $A(T, \bar{\tau})$. In Figure 6, where $\bar{\delta}_1 > 0$, ψ^γ is a facet of $\mathrm{co}(\{\bar{w}^3\} \times \{m\}, \bar{\sigma}^2 \times \{m+1\})$, where $\bar{\sigma}^2$ is given by $\bar{\sigma}^2(\bar{y}^1, \bar{\pi})$ with $\bar{y}^1 = \bar{y}$ and $\bar{\pi} = (Z^0, \pi_1) = (\bar{\omega}_1, \bar{\omega}_2)$, and $\bar{\sigma}^2$ lies in $A(\bar{T}, \bar{\tau})$ with $\bar{T} = \{\bar{\omega}_1, \bar{\omega}_2\}$. This completes the description of the triangulation of the boundary faces $S(U) \times [1, \infty)$, $U \subset I$, of $S \times [1, \infty)$.

6. The steps of the continuous deformation algorithm on S. In this section the steps of the continuous deformation algorithm on S with arbitrary grid refinement are given. So, let $S \times [1, \infty)$ be triangulated as described in §4 for a sequence of decreasing grid sizes d_m^{-1}, $m = 1, 2, \ldots$, such that S is triangulated on level m according to the V-triangulation with grid size d_m^{-1}. In order to find an approximate solution to the NLCP on S with arbitrary accuracy the algorithm follows, by alternating l.p. pivot steps in systems of linear equations and replacement steps in the triangulation of $S \times [1, \infty)$, a piecewise linear path of points satisfying (2.3) or (2.4). On level 1 the paths of points satisfying (2.3) are followed by the steps of the product-ray algorithm described in §3. These paths connect the starting point $(v, 1)$ with a point $(p, 1)$ satisfying (2.4) or two points on level 1 satisfying (2.4). We will now show that the points satisfying (2.4) form piecewise linear paths which can be followed by alternating l.p. pivot steps in a linear system corresponding to (2.4) and replacement steps in the triangulation of $S(U) \times [m, m+1]$ for varying proper subsets U of I and for varying m, $m = 1, 2, \ldots$. A linear piece of such a path corresponds to a line segment of solutions to a system of linear equations with respect to a so-called complete $(n - u + 1)$-simplex ψ^γ in $S(U) \times [m, m+1]$ for some proper subset U of I and for some m, $m \geqslant 1$.

DEFINITION 6.1. Let U be a subset of I, γ_j a permutation of the elements in $I(j) \setminus U_j$, $j \in I_N$, $\gamma = (\gamma_1, \ldots, \gamma_N)$, and let ψ^γ be a k-simplex, $k = n - u, n - u + 1$, in $S(U) \times [m, m+1]$ for certain m, $m = 1, 2, \ldots$. The simplex $\psi^\gamma(x^1, \ldots, x^{k+1})$ with $x^i = (p^i, t_i)$, $p^i \in S$ and $t_i \in \{m, m+1\}$, is complete if the system of linear equations

$$\sum_{i=1}^{k+1} \lambda_i \begin{pmatrix} z(p^i) \\ 1 \end{pmatrix} + \sum_{(i,h) \in U} \mu_{i,h} \begin{pmatrix} e(i,h) \\ 0 \end{pmatrix} - \sum_{j=1}^{N} \beta_j \begin{pmatrix} \bar{e}(j) \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} \tag{6.1}$$

has a solution $\lambda_i^* \geqslant 0$, $i = 1, \ldots, k+1$, $\mu_{i,h}^* \geqslant 0$, $(i, h) \in U$, and β_j^*, $j \in I_N$.

A solution λ_i^*, $i = 1, \ldots, k+1$, $\mu_{i,h}^*$, $(i, h) \in U$, β_j^*, $j \in I_N$ will be denoted by $(\lambda^*, \mu^*, \beta^*)$.

Nondegeneracy assumption. If ψ^γ is a complete k-simplex in $S(U) \times [m, m+1]$ then for $k = n - u$ the system (6.1) has a unique solution $(\lambda^*, \mu^*, \beta^*)$ with $\lambda_i^* > 0$, $i = 1, \ldots, n - u + 1$, $\mu_{i,h}^* > 0$, $(i, h) \in U$, and for $k = n - u + 1$ at most one variable of (λ^*, μ^*) is equal to zero.

By this nondegeneracy assumption, a complete $(n - u + 1)$-simplex ψ^γ contains a line segment of solutions with two end points. An end point is characterized by a solution with exactly one variable in (λ^*, μ^*) equal to zero. All other variables in (λ^*, μ^*) are positive. We call the solution at an end point of such a line segment a basic solution. To each solution of (6.1) there corresponds a point $x = \sum_{i=1}^{n-u+2} \lambda_i^* x^i$ in ψ^γ satisfying (2.4). In particular, when at a basic solution one of the λ_i^*'s, say $\lambda_{\bar{s}}^*$, is equal to zero, the corresponding x lies in the interior of the facet of ψ^γ opposite vertex $x^{\bar{s}}$. This facet is then also complete. If at a basic solution $\mu_{i,h}^* = 0$ for some (i, h) in U, then the corresponding x lies in the interior of ψ^γ. Each line segment of solutions to (6.1) induces in this way a line segment of points x in ψ^γ satisfying (2.4) with two end points. This line segment of points can be followed by making a linear programming step in the system (6.1).

Now let m be a fixed positive integer and let U be a proper subset of I. Under the nondegeneracy assumption the complete $(n - u + 1)$-simplices in $S(U) \times [m, m + 1]$ determine sequences of adjacent simplices with complete facets such that each sequence is either a loop or has two end simplices. An end simplex yields either (1) a complete facet on level m, (2) a complete facet on level $m + 1$, (3) a complete facet in $S(U \cup \{(j, k)\})$ for some $(j, k) \notin U$ or (4), if U is nonempty, a complete $(n - u + 1)$-simplex in $S(U \setminus \{(i, h)\}) \times [m, m + 1]$ with $\mu_{i,h}^* = 0$ for some (i, h) in U. In case (3), the facet is an end simplex of a sequence of complete $(n - u)$-simplices in $S(U \cup \{(j, k)\}) \times [m, m + 1]$, and in case (4) the simplex is a facet of a complete $(n - u + 2)$-simplex in $S(U \setminus \{(i, h)\}) \times [m, m + 1]$ which is an end simplex of a sequence of adjacent complete $(n - u + 2)$-simplices in $S(U \setminus \{(i, h)\}) \times [m, m + 1]$. Linking the sequences of complete simplices in $S(U) \times [m, m + 1]$ in this way together for varying U we obtain sequences of adjacent complete simplices of varying dimension in $S \times [m, m + 1]$. Again each sequence is either a loop or has two end simplices. An end simplex gives either a complete simplex in $S(U) \times \{m\}$ for some $U \subset I$ or a complete simplex in $S(U') \times \{m + 1\}$ for some $U' \subset I$. In the former case, if $m > 1$, the complete simplex is a facet of an end simplex of a sequence of adjacent complete simplices in $S(U) \times [m - 1, m]$, and in the latter case it is a facet of an end simplex of a sequence of adjacent complete simplices in $S(U') \times [m + 1, m + 2]$.

For varying m, $m \geqslant 1$, the complete simplices therefore yield sequences of adjacent complete simplices in $S \times [1, \infty)$. Each sequence has 0, 1 or 2 end simplices. Each end simplex gives a complete simplex in $S \times \{1\}$. A sequence with two end simplices connects therefore two complete simplices in $S \times \{1\}$, while a sequence with one end simplex has a complete simplex in $S \times \{1\}$ and must exceed each level $S \times \{m\}$, $m = 2, 3, \ldots$ since the number of simplices in $S \times [1, m]$ is finite for each m. A sequence with no end simplices is either a loop and remains in $S \times [m_0, m_1]$ for certain $1 \leqslant m_0 < m_1 < \infty$, or there is an m_0, $m_0 \geqslant 1$, for which the path exceeds each level $S \times \{m\}$, $m > m_0$ with at least two different complete simplices.

Since each complete $(n - u + 1)$-simplex in $S(U) \times [m, m + 1]$ yields a line segment of points satisfying (2.4), the complete simplices in $S \times [1, \infty)$ therefore determine, under the nondegeneracy assumption, piecewise linear paths of points satisfying (2.4). Each end point of such a path lies on level 1 and is also an end point of a p.l. path of points satisfying (2.3) and conversely. The p.l. paths of points satisfying (2.3) or (2.4) can therefore be linked to p.l. paths in $S \times [1, \infty)$. Exactly one path, say P, has $(v, 1)$ as end point. There are no other end points. Since the number of simplices and faces up to level m is finite the path P when followed by starting in $(v, 1)$ exceeds each level m within a finite number of pieces, $m = 1, 2, \ldots$. The path P can be followed as follows.

The algorithm starts on level one in the point v with the variable dimension algorithm described in §3 yielding within a finite number of steps a complete simplex τ^0 in $S \times \{1\}$. Then the algorithm continues by following a sequence of adjacent complete simplices in $S \times [1, \infty)$ starting with the unique complete simplex ψ in

$S \times [1,2]$ containing τ^0 as a facet. The algorithm can be terminated when the accuracy of an approximate solution is sufficient, e.g. when β^* is small enough. Each time the path returns to $S \times \{1\}$ with a complete simplex τ^1 we again apply the variable dimension restart algorithm of §3 starting with the complete simplex τ^1. This yields within a finite number of steps another complete simplex τ^2 in $S \times \{1\}$. Then the algorithm continues in $S \times [1, \infty)$ starting with the unique complete simplex ψ in $S \times [1,2]$ containing τ^2 as a facet, etc. Observe that both τ^1 and τ^2 differ from all other complete simplices generated on level one. A complete simplex on level 1 with which the product-ray algorithm terminates is either an n-simplex $\tau(w^1, \omega)$ in $A(\gamma)$ or a $(t-1)$-simplex $\tau(w^1, \omega)$ in $A(\gamma) \cap S(U)$ for some permutation vector γ and some proper subset U of I.

In the first case $\tau \times \{1\}$ is a facet of the $(n+1)$-simplex $\psi^{\bar{\gamma}}$ in $S \times [1,2]$ where $\psi^{\bar{\gamma}}$ is given by $\psi^{\bar{\gamma}} = \text{co}(\tau \times \{1\}, \{c(\tau)\} \times \{2\})$ with $\bar{\gamma}_h = \gamma_h$, $h \neq j_0$, and $\bar{\gamma}_{j_0} = ((j_0, k_0^{j_0}), \ldots, (j_0, k_{t(j_0)}^{j_0}), (j_0, k))$, where (j_0, k) is the unique index in I not in the permutation vector γ. Notice that $A(\bar{\gamma}, j_0)$ is equal to $A(\gamma)$.

In the second case w^1 is given by

$$w^1 = v + q^\gamma(Z^0) + \sum_{(j,k) \in Z} a(j,k) d_1^{-1} q^\gamma(j,k)$$

and $\omega = (\omega_1, \ldots, \omega_{t-1})$ is a permutation of the $t-1$ elements of Z. Since $v = v(U) + \sum_{(i,h) \in U} q^\gamma(i,h)$ the vertex w^1 is equal to

$$w^1 = v(U) + \bar{q}^\gamma(Z^0) + \sum_{(j,k) \in Z} a(j,k) d_1^{-1} q^\gamma(j,k)$$

with $\bar{q}^\gamma(Z^0) = q^\gamma(Z^0) + \sum_{(i,h) \in U} q^\gamma(i,h)$. The $(n-u)$-simplex $\tau(w^1, \omega)$ on level one is a facet of the $(n-u+1)$-simplex ψ^γ in $S(U) \times [1,2]$ where ψ^γ is given by $\text{co}(\tau(\bar{w}^1, \bar{\omega}) \times \{1\}, \{c(\tau)\} \times \{2\})$ with $\tau(\bar{w}^1, \bar{\omega})$ an $(n-u)$-simplex in $A(\gamma, j_0)$. The parameters j_0, $\bar{w}^1$, $\bar{\omega}$ and $\bar{a}$ are determined as follows. Let $\omega_{r(j)} = (j, k_{t(j)}^j)$, $j \in I_N$ with $t(j) \geq 1$. Let $r(j_0) = \max\{r(j) | a(\omega_{r(j)}) = \min_i a(\omega_{r(i)})\}$ and let r be equal to $r(j_0)$. Then the vertex $\bar{w}^1$ of τ is given by $\bar{w}^1 = w^{r+1}$, $\bar{\omega}$ is given by $\bar{\omega} = (\omega_{r+1}, \ldots, \omega_{t-1}, Z^0, \omega_1, \ldots, \omega_r)$, $\bar{a}(Z^0) = d_1 - a(\omega_r) - 1$,

$$\bar{a}(\omega_h) = \begin{cases} a(\omega_h) - a(\omega_r), & h = 1, \ldots, r, \\ a(\omega_h) - a(\omega_r) - 1, & h = r+1, \ldots, t-1. \end{cases}$$

Observe that $\bar{a}(\omega_r)$ is equal to zero. In both cases the algorithm continues in $S \times [1, \infty)$ by making an l.p. pivot step with $(z^{\mathrm{T}}(c(\tau)), 1)^{\mathrm{T}}$ in (6.1).

In $S \times [1, \infty)$ a sequence of adjacent complete simplices inducing a p.l. path of points satisfying (2.4) is generated by alternating l.p. pivoting steps in (6.1) and replacement steps of the triangulation as follows. For some proper subset U of I and for some m, $m \geq 1$, let

$$\psi^\gamma(x^1, \ldots, x^{n-u+2}) = \text{co}\big(\text{co}\big(\{w^i | \omega_i \notin T\}\big) \times \{m\}, \sigma(y^1, \pi(T)) \times \{m+1\}\big)$$

be an arbitrarily generated $(n-u+1)$-simplex of the triangulation of $S(U) \times [m, m+1]$, as described in §5. The line segment of solutions (λ, μ, β) to (6.1) with respect to ψ^γ is followed by making an l.p. pivoting step in (6.1) with either a $\lambda_{\bar{s}}$ for some $\bar{s}$, $1 \leq \bar{s} \leq n-u+2$, or a $\mu_{i,h}$ for some $(i,h) \in U$.

If $\mu_{i,h}$ becomes 0 for some $(i, h) \in U$ the algorithm continues with the unique $(n-u+2)$-simplex $\bar{\psi}^{\bar{\gamma}}$, described in Lemma 5.7, in $S(U \setminus \{(i,h)\}) \times [m, m+1]$ having ψ^{γ} as facet. When ψ^{γ} is the facet of $\bar{\psi}^{\bar{\gamma}}$ opposite the vertex $\bar{y}^p$ on level $m+1$, an l.p. pivot step in (6.1) is made with $(z^{\mathrm{T}}(\bar{y}^p), 1)^{\mathrm{T}}$. When ψ^{γ} is the facet of $\bar{\psi}^{\bar{\gamma}}$ opposite the vertex $\bar{w}^1$ on level m, then an l.p. pivot step in (6.1) is made with $(z^{\mathrm{T}}(\bar{w}^1), 1)^{\mathrm{T}}$. If by the l.p. pivot step in (6.1) with respect to ψ^{γ}, $\lambda_{\bar{s}}$ becomes 0 for some $\bar{s}$, $1 \leqslant \bar{s} \leqslant n-u+2$, then the facet of ψ^{γ} opposite the vertex $x^{\bar{s}}$ is also complete and yields a new complete simplex adjacent to ψ^{γ}. We have to consider four different cases:

(A) $x^{\bar{s}} = (w^s, m)$ for some s, $1 \leqslant s \leqslant n-u+1$, and $x^{\bar{s}}$ is not the only vertex of ψ^{γ} on level m;

(B) $x^{\bar{s}} = (w^s, m)$ for some s, $1 \leqslant s \leqslant n-u+1$, and $x^{\bar{s}}$ is the only vertex of ψ^{γ} on level m, so that the facet of ψ^{γ} opposite $x^{\bar{s}}$ lies in $S \times \{m+1\}$;

(C) $x^{\bar{s}} = (y^p, m+1)$ for some p, $1 \leqslant p \leqslant t+1$, and $x^{\bar{s}}$ is not the only vertex of ψ^{γ} on level $m+1$;

(D) $x^{\bar{s}} = (y^p, m+1)$ for some p, $1 \leqslant p \leqslant t+1$, and $x^{\bar{s}}$ is the only vertex of ψ^{γ} on level $m+1$.

In the cases (A) and (C) the algorithm continues with the uniquely determined complete simplex in $S \times [m, m+1]$ adjacent to ψ^{γ} sharing with ψ^{γ} the facet opposite the vertex $x^{\bar{s}}$ of ψ^{γ}. In case (B) the algorithm continues with the uniquely determined $(n-u+1)$-simplex in $S(U) \times [m+1, m+2]$ also having the facet of ψ^{γ} opposite $x^{\bar{s}}$ on level $m+1$ as a facet, and in case (D) the algorithm continues with the unique $(n-u+1)$-simplex in $S(U) \times [m-1, m]$ also having the facet of ψ^{γ} opposite $x^{\bar{s}}$ on level m as a facet, unless $m = 1$. If $m = 1$ in case (D), the algorithm continues as described in §3 with the complete $(n-u)$-simplex $\tilde{\tau}(\tilde{w}^1, \tilde{\omega})$ being the facet of ψ^{γ} opposite $x^{\bar{s}}$ on level 1. The cases (A)–(D) are now described in detail using the lemmas and tables of the §§4 and 5.

Case A. The point $x^{\bar{s}}$ lies on level m and is not the only vertex of ψ^{γ} on this level, i.e. $x^{\bar{s}}$ is equal to (w^s, m) for some s, $1 \leqslant s \leqslant n-u+1$. According to the definition of $A(T, \tau)$, ω_s cannot be added to the set T iff

$$\omega_{s-1} \notin T \quad \text{and} \quad \delta_s = 0. \tag{6.5}$$

If (6.5) does not hold, T becomes $T \cup \{\omega_s\}$, $\pi(T)$ becomes $(\pi_1, \ldots, \pi_t, \omega_s)$ and the algorithm continues by making an l.p. pivot step with $(z^{\mathrm{T}}(y^{t+1}), 1)^{\mathrm{T}}$ in system (6.1) with y^{t+1} the new vertex of $\sigma(y^1, \pi(T))$. Observe that the other parameters γ, j_0, $\tau(w^1, \omega)$, a, δ, y^1 and R do not change.

If (6.5) holds, then the t-simplex $\sigma(y^1, \pi(T))$ lies in the facet of $\tau(w^1, \omega)$ opposite vertex w^s. In this case we must also adapt τ to determine the new complete simplex adjacent to ψ^{γ}. The following two cases can occur.

(a) the facet of τ opposite vertex w^s lies in the boundary of $A(\gamma, j_0)$,

(b) the facet of τ opposite vertex w^s does not lie in the boundary of $A(\gamma, j_0)$.

In case (a) one of the three cases of Lemma 5.3 occurs:

(1) If $s = 1$, $\omega_1 = Z^0$ and $a(\omega_1) = d_m - 1$, then the facet of τ opposite vertex w^1 lies in $S(U \cup \{(j_0, k^{j_0}_{t(j_0)})\})$ and γ, j_0, τ, a and δ are adapted according to Lemma 5.4. The t-simplex $\sigma(y^1, \pi(T))$ now also lies in $A(T, \bar{\tau})$ with $\bar{\tau}$ the new simplex τ on level m. The algorithm continues by making a pivot step in (6.1) with $(e^{\mathrm{T}}(j_0, k^{j_0}_{t(j_0)}), 0)^{\mathrm{T}}$.

(2) If $1 < s \leqslant n-u+1$, $\omega_s = (j, k^j_i)$ for certain $j \in I_N$, $1 \leqslant i \leqslant t(j)$, $\omega_{s-1} = (j, k^j_{i-1})$ if $i > 1$ and $\omega_{s-1} = Z^0$ if $i = 1$, and $a(\omega_s) = a(\omega_{s-1})$, then the facet of τ opposite vertex w^s lies in the boundary of $A(\bar{\gamma}, j_0)$ with $\bar{\gamma} \neq \gamma$, and γ and $\tau(w^1, \omega)$ are adapted according to Lemma 5.5 case (1). The t-simplex $\sigma(y^1, \pi(T))$ now also lies in $A(T, \bar{\tau})$ with $\bar{\tau}$ the new simplex τ on level m. The algorithm continues

by making a pivot step in (6.1) with $(z^{\mathrm{T}}(\overline{w}^s), 1)^{\mathrm{T}}$, where $\overline{w}^s$ is the new vertex of τ on level m.

(3) If $s = n - u + 1$, $\omega_{n-u} = (j, k^j_{t(j)})$ for certain $j \in I_N$ and $a(\omega_{n-u}) = 0$, then the facet of τ opposite vertex w^{n-u+1} lies in the boundary of $A(\gamma, j)$, and j_0 and $\tau(w^1, \omega)$ are adapted according to Lemma 5.5 case (2). The t-simplex $\sigma(y^1, \pi(T))$ now also lies in $A(T, \bar{\tau})$ with $\bar{\tau}$ the new simplex τ on level m. The algorithm continues by making a pivot step in (6.1) with $(z^{\mathrm{T}}(\overline{w}^{n-u+1}), 1)^{\mathrm{T}}$, where $\overline{w}^{n-u+1}$ is the new vertex of τ on level m.

These three cases conclude case (a).

In case (b) the facet of τ opposite vertex w^s does not lie in the boundary of $A(\gamma, j_0)$, i.e. there is another $(n-u)$-simplex $\bar{\tau}(\overline{w}^1, \overline{\omega})$ in $A(\gamma, j_0)$ such that τ and $\bar{\tau}$ share the facet of τ opposite vertex w^s. In this case $\tau(w^1, \omega)$, a and δ are adapted according to the Tables 5 and 6. Let $\overline{w}$ be the new vertex of τ and $\delta(\overline{w})$ the corresponding δ-coefficient. If $\delta(\overline{w})$ is positive, then T, $\sigma(y^1, \pi(T))$ and R are adapted as follows. T becomes $T \cup \{\omega_{s-1}\}$, $\pi(T)$ becomes $(\omega_{s-1}, \pi_1, \ldots, \pi_t)$ and R becomes $R + (\delta(\overline{w}) - 1)e(\omega_{s-1})$. The new simplex σ lies in $A(\overline{T}, \bar{\tau})$ where $\overline{T}$ and $\bar{\tau}$ are the new set T and the new simplex τ on level m. The algorithm continues by making a pivot step in (6.1) with $(z^{\mathrm{T}}(\bar{y}^1), 1)^{\mathrm{T}}$, where $\bar{y}^1$ is the new vertex of σ. If $\delta(\overline{w})$ is zero, then σ lies in $A(T, \bar{\tau})$ with $\bar{\tau}$ the new simplex τ on level m and the algorithm continues by making a pivot step with $(z^{\mathrm{T}}(\overline{w}), 1)^{\mathrm{T}}$, where $\overline{w}$ is the new vertex of $\bar{\tau}$. This concludes case (b) and case A.

Case B. The vertex $x^{\bar{s}} = (w^s, m)$ is the only vertex of ψ^γ on level m. The $(n-u)$-simplex $\sigma(y^1, \pi(T))$ is a simplex of the triangulation $G_{m+1}(\gamma, j_0)$ of $S(U)$ with $T \subset \{\omega_1, \ldots, \omega_{n-u+1}\}$ and $|T| = n - u$. The algorithm continues with the unique $(n-u+1)$-simplex $\bar{\psi}$ in $S(U) \times [m+1, m+2]$ having σ as a facet on level $m+1$. The $(n-u)$-simplex σ has to be reformulated as an $(n-u)$-simplex $\bar{\tau}(\overline{w}^1, \overline{\omega})$ in $G_{m+1}(\gamma, j_0)$ for which $\bar{\tau} = \sigma$, and $\bar{\psi}$ is the convex hull of $\bar{\tau} \times \{m+1\}$ and its centre point $c(\bar{\tau})$ on level $m+2$. The parameters of $\bar{\tau}$ are determined as follows. Let ω_l be the unique element in the set $\{\omega_1, \ldots, \omega_{n-u+1}\}$ not in T, define π_{n-u+1} by ω_l and let r be the index such that $\pi_r = \omega_{n-u+1}$. The centre point of τ is equal to

$$c(\tau) = w^1 + \sum_{h=1}^{n-u} \alpha_h d_{m+1}^{-1} q^\gamma(\omega_h)$$

where $\alpha_h = \sum_{k=h+1}^{n-u+1} \delta_k$, $h = 1, \ldots, n-u$. Let α_{n-u+1} be equal to zero. Furthermore, the vertex w^1 of τ is equal to

$$w^1 = v(U) + \sum_{h=1}^{n-u} a(\omega_h) d_m^{-1} q^\gamma(\omega_h)$$

$$= v(U) + \sum_{h=1}^{n-u} a(\omega_h) k_m d_{m+1}^{-1} q^\gamma(\omega_h).$$

Combining these two results yields

$$c(\tau) = v(U) + \sum_{h=1}^{n-u} \left(a(\omega_h) k_m + \alpha_h\right) d_{m+1}^{-1} q^\gamma(\omega_h).$$

The vertex y^1 can now be given by

$$
\begin{aligned}
y^1 &= c(\tau) + \sum_{h=1}^{n-u+1} R_{\omega_h} d_{m+1}^{-1} q^{\gamma}(\omega_h) \\
&= v(U) + \sum_{h=1}^{n-u} \left(a(\omega_h)k_m + \alpha_h + R_{\omega_h}\right) d_{m+1}^{-1} q^{\gamma}(\omega_h) \\
&\quad + R_{\omega_{n-u+1}} d_{m+1}^{-1} q^{\gamma}(\omega_{n-u+1}) \\
&= v(U) + \sum_{h=1}^{n-u} \left(a(\omega_h)k_m + \alpha_h + R_{\omega_h} - R_{\omega_{n-u+1}}\right) d_{m+1}^{-1} q^{\gamma}(\omega_h).
\end{aligned}
$$

The parameters of the $(n-u)$-simplex $\bar{\tau}(\bar{w}^1, \bar{\omega})$, being the facet of $\bar{\psi}$ on level $m+1$, are now given by

$$
\bar{w}^1 = y^{r+1} = y^1 - \sum_{h=r+1}^{n-u+1} d_{m+1}^{-1} q^{\gamma}(\pi_h), \quad \bar{\omega} = (\pi_{r+1}, \ldots, \pi_{n-u+1}, \pi_1, \ldots, \pi_r) \quad \text{and}
$$

$$
\bar{a}(\omega_h) = \begin{cases} a(\omega_h)k_m + \alpha_h + R_{\omega_h} - R_{\omega_{n-u+1}}, & \omega_h \notin \{\pi_{r+1}, \ldots, \pi_{n-u+1}\}, \\ a(\omega_h)k_m + \alpha_h + R_{\omega_h} - R_{\omega_{n-u+1}} - 1, & \omega_h \in \{\pi_{r+1}, \ldots, \pi_{n-u+1}\}, \end{cases}
$$

for $h = 1, \ldots, n-u+1$. Observe that $\bar{a}(\omega_{n-u+1})$ is equal to zero and that $\bar{\omega}_{n-u+1}$ is equal to ω_{n-u+1}.

If it is the first time that we move into $S \times [m+1, m+2]$, we choose an integer $k_{m+1} > 1$ and integers $\theta_i^{m+1} \geqslant 0$, $i = 1, \ldots, n+1$, such that $\Sigma_{i=1}^{n+1} \theta_i^{m+1} = k_{m+1}$. In general we should choose θ_i^{m+1}, $i = 1, \ldots, n+1$, in such a way that the centre point $c(\bar{\tau})$ lies close to the approximate solution found on level $m+1$. The intersection of $\bar{\psi}$ and $S(U) \times \{m+2\}$ is $(c(\bar{\tau}), m+2)$ and the algorithm continues by setting τ equal to $\bar{\tau}$, $T = \varnothing$, $y^1 = c(\bar{\tau})$, $R = 0$, $\sigma = \{y^1\}$, $m = m+1$ and by making a pivot step in (6.1) with $(z^{\mathrm{T}}(y^1), 1)^{\mathrm{T}}$. Notice that γ and j_0 do not change. This concludes case B.

Case C. The vertex $x^{\bar{s}}$ lies on level $m+1$ and is not the only vertex of ψ^{γ} on level $m+1$, i.e. $x^{\bar{s}}$ is equal to $(y^p, m+1)$ for some p, $1 \leqslant p \leqslant t+1$. In this case we consider the following two cases

(a) the facet of $\sigma(y^1, \pi(T))$ opposite vertex y^p lies in the boundary of $A(T, \tau)$,

(b) the facet of $\sigma(y^1, \pi(T))$ opposite vertex y^p does not lie in the boundary of $A(T, \tau)$.

In case (a) one of the three cases of Lemma 4.1 occurs:

(1) If $p = 1$, $\delta_s - R_{\omega_s} = 1$ with $\omega_s = \pi_1$, and $\omega_{s-1} \notin T$, the facet of $\sigma(y^1, \pi(T))$ opposite vertex y^1 lies in the facet of $\tau(w^1, \omega)$ opposite vertex w^s. This facet of τ either lies in the boundary of $A(\gamma, j_0)$ or not. First consider the case that the facet of τ opposite vertex w^s lies in the boundary of $A(\gamma, j_0)$, then one of the three cases of Lemma 5.3 holds.

(i) If $s = 1$, $\omega_1 = Z^0$ and $a(\omega_1) = d_m - 1$, then the facet of τ opposite vertex w^1 lies in $S(U \cup \{(j_0, k_{t(j_0)}^{j_0})\})$ and γ, j_0, $\tau(w^1, \omega)$, a and δ are adapted according to Lemma 5.4. The facet of σ opposite vertex y^1 becomes the new simplex σ, i.e. T becomes $T \setminus \{\omega_1\}$, y^1 becomes y^2, $\pi(T)$ becomes $(\pi_2, \ldots, \pi_t)$ and R becomes $R - R_{\omega_1} e(\omega_1)$. The new simplex σ lies in $A(\bar{T}, \bar{\tau})$ where $\bar{T}$ and $\bar{\tau}$ are the new subset T and the new simplex τ on level m. The algorithm continues by making a pivot step in (6.1) with $(e^{\mathrm{T}}(j_0, k_{t(j_0)}^{j_0}), 0)^{\mathrm{T}}$.

(ii) If $1 < s \leqslant n - u + 1$, $\omega_s = (j, k_i^j)$ for certain $j \in I_N$, $1 \leqslant i \leqslant t(j)$, $\omega_{s-1} = (j, k_{i-1}^j)$ if $i > 1$ and $\omega_{s-1} = Z^0$ if $i = 1$, and $a(\omega_{s-1}) = a(\omega_s)$, then the facet of τ opposite vertex w^s lies also in the boundary of $A(\bar{\gamma}, j_0)$ with $\bar{\gamma} \neq \gamma$, and γ and $\tau(w^1, \omega)$ are adapted according to Lemma 5.5 case (1). The parameters T, $\sigma(y^1, \pi(T))$ and R are adapted as follows. T becomes $T \setminus \{\omega_s\} \cup \{\bar{\omega}_s\}$, where $\bar{\omega}_s$ is the sth component of the new permutation vector ω, y^1 becomes $y^2 - d_{m+1}^{-1} q^{\bar{\gamma}}(\bar{\omega}_s)$, $\pi(T)$ becomes $(\bar{\omega}_s, \pi_2, \ldots, \pi_t)$ and R becomes $R + R_{\omega_s}(e(\bar{\omega}_s) - e(\omega_s))$. The algorithm continues by making a pivot step in (6.1) with $(z^{\mathrm{T}}(\bar{y}^1), 1)^{\mathrm{T}}$, where $\bar{y}^1$ is the new vertex of σ on level $m + 1$.

(iii) If $s = n - u + 1$, $\omega_{n-u} = (j, k_{t(j)}^j)$ for certain $j \in I_N$ and $a(\omega_{n-u}) = 0$, then the facet of τ opposite vertex w^{n-u+1} lies also in the boundary of $A(\gamma, j)$ and j_0 and $\tau(w^1, \omega)$ are adapted according to Lemma 5.5 case (2). The parameters T, $\sigma(y^1, \pi(T))$ and R are adapted as follows. T becomes $T \setminus \{\omega_{n-u+1}\} \cup \{\omega_{n-u}\}$, y^1 becomes $y^2 - d_{m+1}^{-1} q^{\gamma}(\omega_{n-u})$, $\pi(T)$ becomes $(\omega_{n-u}, \pi_2, \ldots, \pi_t)$ and R becomes $R - R_{\omega_{n-u+1}}(e(\omega_{n-u+1}) - e(\omega_{n-u}))$. The algorithm continues by making a pivot step in (6.1) with $(z^{\mathrm{T}}(\bar{y}^1), 1)^{\mathrm{T}}$, where $\bar{y}^1$ is the new vertex of σ on level $m + 1$.

This concludes the case that the facet of τ opposite vertex w^s lies in the boundary of $A(\gamma, j_0)$. If this facet does not lie in the boundary of $A(\gamma, j_0)$, then $\tau(w^1, \omega)$ and δ are adapted according to Tables 5 and 6. Let $\bar{w}$ be the new vertex of τ and $\delta(\bar{w})$ the corresponding δ-coefficient of the vertex $\bar{w}$. If $\delta(\bar{w})$ is positive, then T, $\sigma(y^1, \pi(T))$ and R are adapted as follows. T becomes $T \setminus \{\omega_s\} \cup \{\omega_{s-1}\}$, y^1 becomes $y^2 - d_{m+1}^{-1} q^{\gamma}(\omega_{s-1})$, $\pi(T)$ becomes $(\omega_{s-1}, \pi_2, \ldots, \pi_t)$, and R becomes $R - R_{\omega_s} e(\omega_s) + (\delta(\bar{w}) - 1) e(\omega_{s-1})$. The algorithm continues by making a pivot step in (6.1) with $(z^{\mathrm{T}}(\bar{y}^1), 1)^{\mathrm{T}}$ where $\bar{y}^1$ is the new vertex on level $m + 1$. If $\delta(\bar{w})$ is equal to zero, then T becomes $T \setminus \{\omega_s\}$, y^1 becomes y^2, $\pi(T)$ becomes $(\pi_2, \ldots, \pi_t)$, and R becomes $R - R_{\omega_s} e(\omega_s)$. The algorithm continues by making a pivot step in (6.1) with $(z^{\mathrm{T}}(\bar{w}), 1)^{\mathrm{T}}$. This concludes the description of case (1).

(2) If $1 < p < t + 1$, $\delta_s - R_{\omega_s} + R_{\omega_{s-1}} = 0$ with $\omega_s = \pi_p$, and $\omega_{s-1} = \pi_{p-1}$, then the facet of $\sigma(y^1, \pi(T))$ opposite vertex y^p lies in the facet of $\tau(w^1, \omega)$ opposite vertex w^s. This facet of τ again either lies in the boundary of $A(\gamma, j_0)$ or not. First consider the case that the facet of τ opposite vertex w^s lies in the boundary of $A(\gamma, j_0)$, then one of the three cases of Lemma 5.3 holds.

(i) If $s = 1$, $\omega_1 = Z^0$ and $a(\omega_1) = d_m - 1$, then the facet of τ opposite vertex w^1 lies in $S(U \cup \{(j_0, k_{t(j_0)}^{j_0})\})$ and γ, j_0, $\tau(w^1, \omega)$, a and δ are adapted according to Lemma 5.4. The facet of σ opposite vertex y^p is the new simplex σ. The parameters T, $\sigma(y^1, \pi(T))$ and R are given by $T = T \setminus \{\omega_{n-u+1}\}$, $\pi(T) = (\pi_1, \ldots, \pi_{p-2}, \pi_p, \ldots, \pi_t)$ and $R = R - \delta_1 e(\omega_1) - R_{\omega_{n-u+1}} e(\omega_{n-u+1})$. The algorithm continues by making a pivot step in (6.1) with $(e^{\mathrm{T}}(j_0, k_{t(j_0)}^{j_0}), 0)^{\mathrm{T}}$.

(ii) If $1 < s \leqslant n - u + 1$, $\omega_s = (j, k_i^j)$ for certain $j \in I_N$, $1 \leqslant i \leqslant t(j)$, $\omega_{s-1} = (j, k_{i-1}^j)$ if $i > 1$ and $\omega_{s-1} = Z^0$ if $i = 1$, and $a(\omega_{s-1}) = a(\omega_s)$, then the facet of τ opposite vertex w^s lies also in the boundary of $A(\bar{\gamma}, j_0)$ with $\bar{\gamma} \neq \gamma$, and γ and $\tau(w^1, \omega)$ are adapted according to Lemma 5.5 case (1). The parameters T, $\sigma(y^1, \pi(T))$ and R are adapted as follows. T becomes $T \setminus \{\omega_{s-1}, \omega_s\} \cup \{\bar{\omega}_{s-1}, \bar{\omega}_s\}$, where $\bar{\omega}$ denotes the new permutation vector, $\pi(T)$ becomes $(\pi_1, \ldots, \pi_{p-2}, \bar{\omega}_{s-1}, \bar{\omega}_s, \pi_{p+1}, \ldots, \pi_t)$ and R becomes $R + R_{\omega_{s-1}}(e(\bar{\omega}_{s-1}) - e(\omega_{s-1})) + R_{\omega_s}(e(\bar{\omega}_s) - e(\omega_s))$. The algorithm continues by making a pivot step in (6.1) with $(z^{\mathrm{T}}(\bar{y}^p), 1)^{\mathrm{T}}$, where $\bar{y}^p$ is the new vertex of σ on level $m + 1$.

(iii) If $s = n - u + 1$, $\omega_{n-u} = (j, k_{t(j)}^j)$ for certain $j \in I_N$ and $a(\omega_{n-u}) = 0$, then the facet of τ opposite vertex w^{n-u+1} lies also in the boundary of $A(\gamma, j)$, and j_0 and $\tau(w^1, \omega)$ are adapted according to Lemma 5.5 case (2). The parameters $\sigma(y^1, \pi(T))$ and R are adapted as follows, $\pi(T)$ becomes $(\pi_1, \ldots, \pi_{p-2}, \pi_p, \pi_{p-1}, \ldots, \pi_t)$ and R becomes $R - (R_{\omega_{n-u+1}} - R_{\omega_{n-u}})(e(\omega_{n-u}) - e(\omega_{n-u+1}))$. The algorithm continues by making a pivot step in (6.1) with $(z^{\mathrm{T}}(\bar{y}^p), 1)^{\mathrm{T}}$ where $\bar{y}^p$ is the new vertex of σ on level $m+1$.

This concludes the case that the facet of τ opposite vertex w^s lies in the boundary of $A(\gamma, j_0)$. If this facet does not lie in the boundary of $A(\gamma, j_0)$, then $\tau(w^1, \omega)$ and δ are adapted according to Tables 5 and 6. Let $\bar{w}$ be the new vertex of τ and $\delta(\bar{w})$ the corresponding δ-coefficient. The parameters $\sigma(y^1, \pi(T))$ and R are adapted as follows: $\pi(T)$ becomes $(\pi_1, \ldots, \pi_{p-2}, \pi_p, \pi_{p-1}, \ldots, \pi_t)$ and R becomes $R - \delta_s e(\omega_s) + \delta(\bar{w})e(\omega_{s-1})$. The algorithm continues by making a pivot step in (6.1) with $(z^{\mathrm{T}}(\bar{y}^p), 1)^{\mathrm{T}}$ where $\bar{y}^p$ is the new vertex of σ on level $m+1$. This concludes the description of case (2).

(3) If $p = t+1$ and $R_{\omega_s} = 0$ with $\omega_s = \pi_t$, then the facet of $\sigma(y^1, \pi(T))$ opposite vertex y^{t+1} lies in $A(T \setminus \{\omega_s\}, \tau)$. The parameters T and $\sigma(y^1, \pi(T))$ are adapted as follows. T becomes $T \setminus \{\omega_s\}$ and $\pi(T)$ becomes $(\pi_1, \ldots, \pi_{t-1})$. The algorithm continues by making a pivot step in (6.1) with $(z^{\mathrm{T}}(w^s), 1)^{\mathrm{T}}$. This concludes case (3) and furthermore it concludes the case that the facet of $\sigma(y^1, \pi(T))$ opposite vertex y^p lies in the boundary of $A(T, \tau)$.

Now consider case (b) that the facet of $\sigma(y^1, \pi(T))$ opposite vertex y^p does not lie in the boundary of $A(T, \tau)$. In this case $\sigma(y^1, \pi(T))$ and R are adapted according to Table 2. All other parameters do not change and the algorithm continues by making a pivot step in (6.1) with $(z^{\mathrm{T}}(\bar{y}), 1)^{\mathrm{T}}$, where $\bar{y}$ is the new vertex of σ on level $m+1$. This concludes Case C.

Case D. The vertex $x^{\bar{s}} = (y^p, m+1)$ is the only vertex of ψ^γ on level $m+1$. In this case we have $T = \varnothing$, $t = 0$, and $p = 1$. We will first consider the case that $m > 1$. The $(n-u)$-simplex $\tau(w^1, \omega)$ is a complete simplex on level m of $G_m(\gamma, j_0)$. We have to determine the unique $(n-u)$-simplex $\bar{\tau}(\bar{w}^1, \bar{\omega})$ in $G_{m-1}(\gamma, j_0)$ and the set $\bar{T} \subset \{\bar{\omega}_1, \ldots, \bar{\omega}_{n-u+1}\}$, $|\bar{T}| = n-u$, such that $\tau(w^1, \omega)$ lies in $\bar{\tau}(\bar{w}^1, \bar{\omega})$ and $\tau = \bar{\sigma}(\bar{y}^1, \bar{\pi}(\bar{T}))$ lies in $A(\bar{T}, \bar{\tau})$. This is accomplished as follows. The vertex w^1 is given by

$$\begin{aligned} w^1 &= v(U) + \sum_{h=1}^{n-u} a(\omega_h) d_m^{-1} q^\gamma(\omega_h) \\ &= v(U) + \sum_{h=1}^{n-u} \left(a(\omega_h) k_{m-1}^{-1}\right) d_{m-1}^{-1} q^\gamma(\omega_h). \end{aligned}$$

Let $\bar{a}(\omega_h)$ be the entier of $a(\omega_h)k_{m-1}^{-1}$, for all h, where entier of x, $x \in \mathbb{R}$, is the largest integer less than or equal to x, then $\bar{a}(\cdot)$ satisfies the condition $0 \leqslant \bar{a}(j, k_{t(j)}^j) \leqslant \ldots \leqslant \bar{a}(j, k_1^j) \leqslant \bar{a}(Z^0) \leqslant d_{m-1} - 1$, for all $j \in I_N$. Let $\bar{w}^1$ be given by $\bar{w}^1 = v(U) + \sum_{h=1}^{n-u} \bar{a}(\omega_h) d_{m-1}^{-1} q^\gamma(\omega_h)$, then $\bar{w}^1$ lies in $A(\gamma, j_0)$. Let x be an interior point in τ, then x will also be an interior point in $\bar{\tau}$. The vector x can be written as $x = \sum_{h=1}^{n-u+1} \lambda_h w^h$, with $\lambda_h > 0$, $h = 1, \ldots, n-u+1$ and $\sum_{h=1}^{n-u+1} \lambda_h = 1$. This can be rewritten to

$$x = w^1 + \sum_{h=1}^{n-u} \lambda'_h d_{m-1}^{-1} q^\gamma(\omega_h) \tag{6.6}$$

where $\lambda'_h = 1 - \sum_{i=1}^{h} \lambda_i$, $h = 1, \ldots, n-u$, so that $\lambda'_1 > \lambda'_2 > \ldots > \lambda'_{n-u}$. This gives

$$\begin{aligned} x &= \bar{w}^1 + \sum_{h=1}^{n-u} \left(a(\omega_h) k_{m-1}^{-1} - \bar{a}(\omega_h) + \lambda'_h\right) d_{m-1}^{-1} q^\gamma(\omega_h) \\ &= \bar{w}^1 + \sum_{h=1}^{n-u} b(\omega_h) d_{m-1}^{-1} q^\gamma(\omega_h). \end{aligned}$$

According to (6.6) x is also given by $x = \bar{w}^1 + \sum_{h=1}^{n-u}\beta_h d_{m-1}^{-1} q^{\gamma}(\bar{\omega}_h)$ for $\beta_1 > \beta_2 > \dots > \beta_{n-u}$ where $\bar{\omega} = (\bar{\omega}_1, \dots, \bar{\omega}_{n-u+1})$ is given by $\bar{\omega}_{n-u+1} = \omega_{n-u+1}$ and $b(\bar{\omega}_1) > b(\bar{\omega}_2) > \dots > b(\bar{\omega}_{n-u})$.

Now we have determined the $(n-u)$-simplex $\bar{\tau}(\bar{w}^1, \bar{\omega})$ in $G_{m-1}(\gamma, j_0)$, we still have to determine the representation of $\tau(w^1, \omega)$ as a t-simplex $\bar{\sigma}(\bar{y}^1, \bar{\pi}(\bar{T}))$ in $A(\bar{T}, \bar{\tau})$. The set $\bar{T}$ is a subset of $\{\bar{\omega}_1, \dots, \bar{\omega}_{n-u+1}\}$ containing $n-u$ elements, i.e. $t = n-u$. The centre point of $\bar{\tau}$ is given by

$$c(\bar{\tau}) = \sum_{h=1}^{n-u+1} \delta_h k_{m-1}^{-1} \bar{w}^h = \bar{w}^1 + \sum_{h=1}^{n-u} \bar{\alpha}_h d_m^{-1} q^{\gamma}(\bar{\omega}_h)$$

$$= \bar{w}^1 + \sum_{h=1}^{n-u} \alpha_h d_m^{-1} q^{\gamma}(\omega_h)$$

for certain coefficients $(\bar{\alpha}_h)_{h=1}^{n-u}$ and $(\alpha_h)_{h=1}^{n-u}$, where $\bar{\alpha}_h = \sum_{i=h+1}^{n-u+1}\delta_i$ and $\alpha_h = \bar{\alpha}_l$ when $\omega_h = \bar{\omega}_l$. Furthermore we have

$$w^1 - \bar{w}^1 = \sum_{h=1}^{n-u} \left(a(\omega_h) - \bar{a}(\omega_h) k_{m-1}\right) d_m^{-1} q^{\gamma}(\omega_h)$$

so that

$$w^1 = \bar{w}^1 + (w^1 - \bar{w}^1)$$

$$= c(\bar{\tau}) - \sum_{h=1}^{n-u} \alpha_h d_m^{-1} q^{\gamma}(\omega_h) + \sum_{h=1}^{n-u} \left(a(\omega_h) - \bar{a}(\omega_h) k_{m-1}\right) d_m^{-1} q^{\gamma}(\omega_h)$$

$$= c(\bar{\tau}) + \sum_{h=1}^{n-u} c(\omega_h) d_m^{-1} q^{\gamma}(\omega_h) \tag{6.7}$$

where $c(\omega_h) = a(\omega_h) - \bar{a}(\omega_h) k_{m-1} - \alpha_h$, $h = 1, \dots, n-u$. Let $c(\omega_{n-u+1})$ be equal to zero and $\bar{c}$ the minimum of $c(\omega_h)$, $h = 1, \dots, n-u+1$, then $c(\omega_h) - \bar{c} \geq 0$ for all h. Observe that $\bar{c} \geq 0$. The equation (6.7) can now be expressed as

$$w^1 = c(\bar{\tau}) + \sum_{h=1}^{n-u+1} \left(c(\omega_h) - \bar{c}\right) d_m^{-1} q^{\gamma}(\omega_h)$$

$$= c(\bar{\tau}) + \sum_{h=1}^{n-u+1} \bar{c}(\omega_h) d_m^{-1} q^{\gamma}(\omega_h).$$

Let s be the index given by $s = \max\{i \in I_{n-u+1} | \bar{c}(\omega_i) = 0\}$, i.e., $\bar{c}(\omega_{s+1}) > 0, \dots, \bar{c}(\omega_{n-u+1}) > 0$, then $\bar{T}$ is given by $\bar{T} = \{\omega_1, \dots, \omega_{s-1}, \omega_{s+1}, \dots, \omega_{n-u+1}\}$, and the parameters of $\bar{\sigma}(\bar{y}^1, \bar{\pi}(\bar{T}))$ and $\bar{R}$ are given by $\bar{y}^1 = w^{s+1}$, $\bar{\pi}(\bar{T}) = (\omega_{s+1}, \dots, \omega_{n-u+1}, \omega_1, \dots, \omega_{s-1})$ and

$$\bar{R}_{\omega_h} = \begin{cases} \bar{c}(\omega_h), & h = 1, \dots, s, \\ \bar{c}(\omega_h) - 1, & h = s+1, \dots, n-u+1. \end{cases}$$

Observe that $\bar{R}_{\omega_s}$ is equal to zero. From the construction it is clear that $\bar{\sigma}(\bar{y}^1, \bar{\pi}(\bar{T}))$ lies in $A(\bar{T}, \bar{\tau})$ and that $\bar{\sigma} = \tau$. Let $\bar{s}$ be the index such that $\bar{\omega}_{\bar{s}} = \omega_s$, then the algorithm continues by making a pivot step in (6.1) with $(z^{\mathrm{T}}(\overline{w}^{\bar{s}}), 1)^{\mathrm{T}}$. This concludes the case D for $m > 1$.

In the case $m = 1$ we have found a complete $(n - u)$-simplex $\tau(w^1, \omega)$ in $G_1(\gamma, j_0)$ on level one. In the case $u = 0$, the n-simplex τ is the I-complete n-simplex $\tau(w^1, \omega)$ in $G(\bar{\gamma})$ with $\bar{\gamma}_h = \gamma_h$, $h \neq j_0$, and $\bar{\gamma}_{j_0} = ((j_0, k_0^{j_0}), \ldots, (j_0, k_{t(j_0)-1}^{j_0}))$. The algorithm now continues with the steps of the product-ray algorithm starting in step 4 by performing a pivot step with $(e^{\mathrm{T}}(p), 0)^{\mathrm{T}}$, $p = (j_0, k_{t(j_0)}^{j_0})$, in the system (3.1) as described in §3.

In the case $u > 0$, i.e. U is nonempty, the $(n - u)$-simplex $\tau(w^1, \omega)$ is an $I \setminus U$-complete $(t - 1)$-simplex τ in $S(U)$, where $t = |I \setminus U| - N + 1 = (N + n - u) - N + 1 = n - u + 1$. The $(t - 1)$-simplex τ is a facet of the t-simplex $\bar{\tau}(\overline{w}^1, \bar{\omega})$ in $A(\gamma)$. The parameters of $\bar{\tau}$ are determined as follows. Let r be the index such that $\omega_r = Z^0$, then the vertex w^1 is given by

$$w^1 = v(U) + \sum_{h=1}^{n-u+1} a(\omega_h) d_1^{-1} q^{\gamma}(\omega_h)$$

$$= v + \bar{q}^{\gamma}(Z^0) + \sum_{\substack{h=1 \\ h \neq r}}^{n-u+1} [d_1 + a(\omega_h)] d_1^{-1} q^{\gamma}(\omega_h) + a(\omega_r) d_1^{-1} q^{\gamma}(\omega_r)$$

$$= v + d_1 d_1^{-1} \bar{q}^{\gamma}(Z^0) + \sum_{\substack{h=1 \\ h \neq r}}^{n-u+1} [d_1 - a(\omega_r) + a(\omega_h)] d_1^{-1} q^{\gamma}(\omega_h)$$

where $\bar{q}^{\gamma}(Z^0) = p(Z^0) - v$. The vertex $\overline{w}^1$ is now given by $\overline{w}^1 = w^{r+1} - d_1^{-1} \bar{q}^{\gamma}(Z^0)$, the permutation $\bar{\omega} = (\bar{\omega}_1, \ldots, \bar{\omega}_t)$ is given by $\bar{\omega} = (\omega_r, \ldots, \omega_{n-u+1}, \omega_1, \ldots, \omega_{r-1})$ and the vector $\bar{a}$ is given by $\bar{a}(\omega_h) = d_1 - a(\omega_r) + a(\omega_h) - 1$, $h = 1, \ldots, r$ and $\bar{a}(\omega_h) = d_1 - a(\omega_r) + a(\omega_h)$, $h = r + 1, \ldots, n - u + 1$. From the construction it is clear that τ is the facet of $\bar{\tau}$ opposite vertex $\overline{w}^1$ and that $\bar{\tau}$ lies in $A(\gamma)$. Observe that the set Z is equal to $\{\omega_1, \ldots, \omega_{r-1}, \omega_{r+1}, \ldots, \omega_{n-u+1}\}$, containing $t - 1$ elements. The algorithm now continues with the steps of the product-ray algorithm starting in step 1 by performing a pivot step with $(z^{\mathrm{T}}(\overline{w}^1), 1)^{\mathrm{T}}$ in the system (3.1) as described in §3. This concludes case D for $m = 1$.

The cases above describe the steps of the algorithm to follow a path of complete simplices in $S \times [1, \infty)$ to solve the nonlinear complementarity problem on S with respect to a continuous function from S into R^{N+n}. The algorithm can easily be adapted to follow a path of approximating solutions with respect to a continuous function z from $S \times [1, \infty)$ into R^{N+n} where t, $t \geqslant 1$, can be interpreted as a time parameter. In this case we can apply the algorithm for a constant grid size on each level by taking k_m equal to one, for $m = 1, 2, \ldots$.

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